考试. 闭卷.

平时作业.

可能掉级.

---

主要内容. 环. 模.

详细: Localization.

integral extension.

DDK

dimension.

---

flat

A-CM-ring

regular ring.

同调代数：变形，正合列，

Tor, Ext

reference.

Atiyah.

Eisenbud.

交换环论.

例子：不变量理论.

·代数几何 / 数论.

PDK.

主要技术：局部化

---

$C(X) = \{ f : X \to \mathbb{C} \mid f \text{ continuous} \}.$

$A$ ring $\Rightarrow$ $\operatorname{Spec} A$

$C(X)$, $x \in C(X)$.

$m_x = \{ f \mid f(x) = 0 \}$,

$C(X) / m_x = \mathbb{C}$

$X \to \mathrm{Spec}_m(X) := \{ m \mid m \text{ maximal ideal} \}.$

$x \mapsto m_x.$

In many cases, this is a bijection.

---

$\mathbb{C}^n$, $\mathbb{C}[X_1, \cdots, X_n]$

by Hilbert's Nullstellensatz.

$\mathbb{C}^n \to \mathrm{Spec}_m \mathbb{C}[X_1, \cdots, X_n]$

$$\begin{array}{ccc} a & \to & (X_1 - a_1, \cdots, X_n - a_n). \\ \parallel & & \\ (a_1, \cdots, a_n) & & \end{array}$$

bijection.

---

$X$ $\qquad$ $A = C(X).$

$$x \rightsquigarrow m_x$$

$f \in A$, $\underset{m_x}{\overset{\shortparallel}{m}} \in \operatorname{Spec}_m A$

$$f(x) = f_m$$

$$\left( \begin{array}{c} A \longrightarrow \mathbb{C} \\ f \longmapsto f(x) \end{array} \right).$$

$$\Leftrightarrow \quad A \longrightarrow A/m_x \xrightarrow{\sim} \mathbb{C}$$

$$f \longmapsto \bar{f} \longmapsto \bar{f}(x).$$

homomorphism of $\mathbb{C}$ algebra.

A环. .

$\operatorname{Specm} A = \{ m \mid m \text{ maximal} \}$.

$f \in A$

$f(m) := \bar{f} \in k(m) = A/m$

Why we usally consider

Spec but not Specm?

$X \xrightarrow{\varphi} Y$

continuous, induces.

$C(X) \xleftarrow{\varphi^\#} C(Y)$

$y \circ \varphi \longleftarrow y$

$$B \leftarrow A$$

$$\mathrm{Spec}_m B \longrightarrow \mathrm{Spec}_m A$$

$$m \longrightarrow \varphi^{\#-1}(m).$$

$$X \xrightarrow{\varphi} Y$$

$$x \longrightarrow \varphi(x) = y.$$

$$y \in C(Y) \quad f(y) = 0 \Leftrightarrow \varphi^{\#}(f)(x) = 0.$$

$$\Rightarrow \quad f \in m_y \Leftrightarrow \varphi^{\#}(f) \in m_x.$$

Key question:.

inverse image of maximal

ideal not alway maximal!

---

$$A \xrightarrow{\varphi} B$$

induce $A/\varphi^{-1}(I) \hookrightarrow B/I$

$\Rightarrow$ inverse image of prime $B$ prime.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A/\varphi^{-1}(I) \arrow[r] & B/I \end{tikzcd}$$

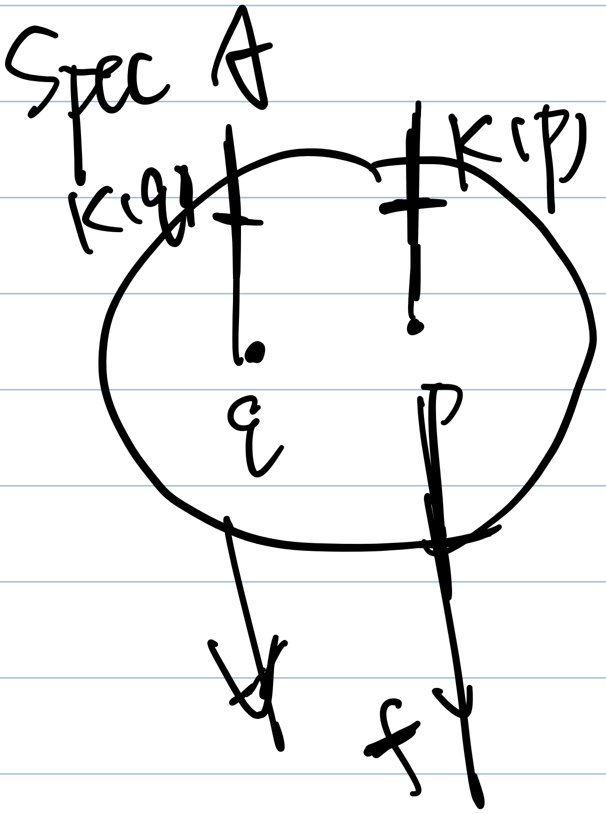
$\operatorname{Spec} A = \{ P \mid P \text{ prime} \}$,

$$A \xrightarrow{f} B$$

$$\operatorname{Spec} A \xleftarrow{\varphi^*} \operatorname{Spec} B$$

$f \in A \qquad \mathfrak{p} \in \operatorname{Spec} A$

$$f(\mathfrak{p}) = \bar{f} \in k(\mathfrak{p}) = \operatorname{Frac}(A/\mathfrak{p})$$

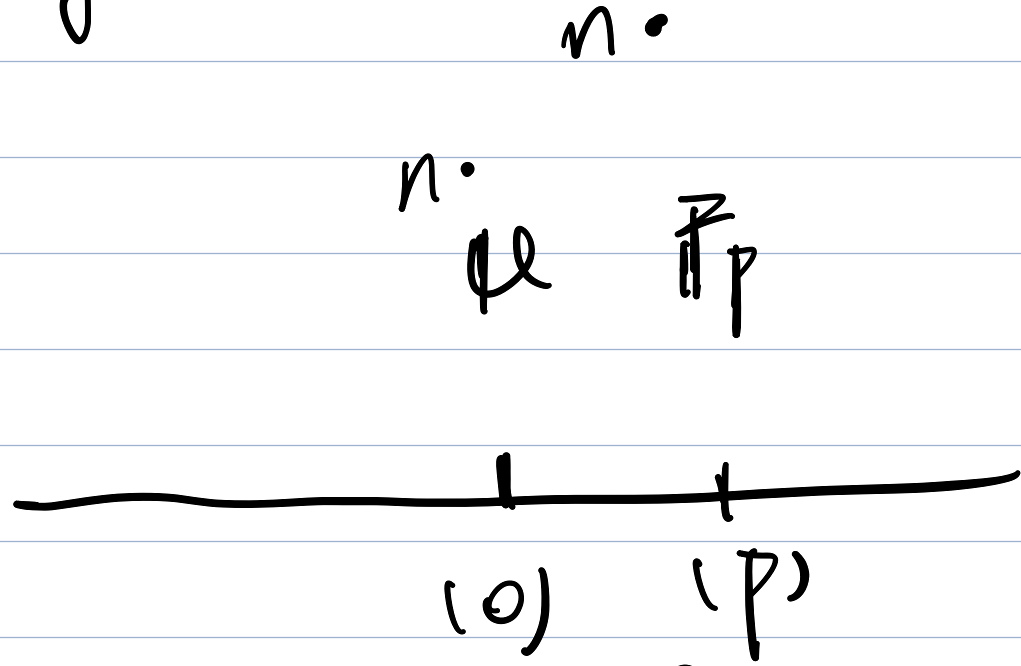



Spec B.

$\boxed{f \text{ is a collection of some functions.}}$

$$f \leftrightarrow (f_{\mathfrak{p}} : \operatorname{Spec} A \to k(\mathfrak{p}))_{\mathfrak{p}}$$

e.g. $\mathbb{Z}$




$$A \xrightarrow{\varphi} B$$

$$f \mapsto \varphi(f)$$

$$\mathrm{Spec}\,A \xleftarrow{\varphi^\#} \mathrm{Spec}\,B$$

$$\varphi^{-1}(p) \longleftarrow p$$

$\kappa(p)$. $\qquad$ $\kappa(\varphi^{-1}(p))$.

$\Downarrow \qquad \Downarrow$

$\mathbb{Q}$: $\varphi(f)(p) \overset{?}{=} f(\varphi^{-1}(p))$ ?

X.

$$\mathrm{Frac}(A/\varphi^{-1}(p)) \to \mathrm{Frac}(B/p)$$

$$\bar{f} \to \overline{\varphi(f)}$$

# Zariski Topology

WANT: $f$ seem as "function" of $\operatorname{Spec} A$ is continuous, $f \notin P$.

$$\Updownarrow$$

$D(f) = \{p \in \operatorname{Spec} A \mid f(p) \neq 0\}$ open.

Property.

$$D(f) \cap D(g) = D(fg).$$

So $\{D(f)\}$ form a topology basis.

Definition. Zariski topology is the topology generated by $\{D(f)\}_{f\in A}$.

glueing $f_p$, let $f$ be a continuous function over Spec $A$.

affine. scheme.

$U \subseteq \operatorname{Spec} A$ open

$\Leftrightarrow$ $U = D(I)$.

$U \subseteq \operatorname{Spec} A$ closed

$\Leftrightarrow$ $U = \bigcap D(f_i)^c$

vanishing points of some $f_i$

$\Leftrightarrow$ $U = V(I) := \{p \mid I \subseteq p\}$.

---

$\phi = V(1)$. $\phi = D(0)$.

$$\bigcap_{i \in I} V(I_i) = V\left(\sum_{i \in I} I_i\right)$$

$$V(I) \cup V(J) = V(IJ).$$

$$D(f) \cap D(g) = D(fg).$$

---

$A$. $\operatorname{Spec} A$.

$f \in A$ $\quad f(p) = \bar{f} \in k(p)$.

$$\operatorname{Spec} A = \bigcup_{i \in I} D(f_i).$$

$\iff V(\sum f_i) = \phi$

$\iff (\sum f_i) = (1)$.

Corollary. Spec $A$ is compact.

Example. $\operatorname{Spec} \mathbb{Z} = \{(0), (2), \cdots\}$.

$(0)$ is not a closed point.

In algebraic geometry, this is called generic point.

Ex. $\mathfrak{p} \in \operatorname{Spec} A$.

then $\{\mathfrak{p}\}$ is closed

$\Leftrightarrow$ $\mathfrak{p}$ is maximal

Question: What are "functions" over $D(f)$?

Answer: Localization!

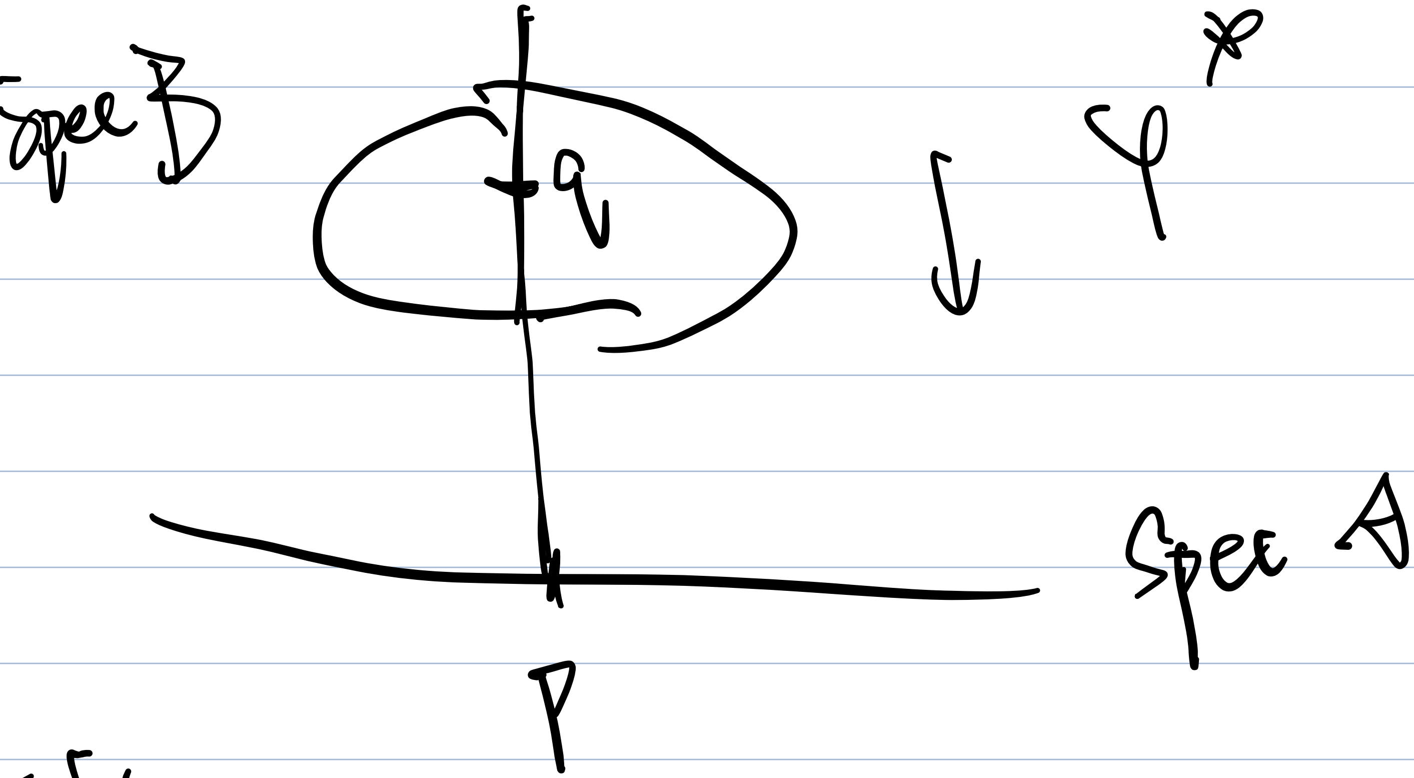
$$D(f) := \operatorname{Spec} A_f$$

universal property.

---

Prop. $\varphi: A \to B$. induce $\varphi^*$

then $\varphi^{*-1}(\mathfrak{p}) \xrightarrow{\sim} \operatorname{Spec} \dfrac{B_{\mathfrak{p}}}{\mathfrak{p} B_{\mathfrak{p}}}$




pf:

$$\varphi^*(q) = p$$

$$\Leftrightarrow \quad q \supseteq \varphi(p)$$

$$\text{and} \quad \varphi^{-1}(q) \subseteq p$$

---

$$\bigcap_{p \in \operatorname{Spec} A} p = \sqrt{(0)}.$$

$$\Rightarrow \bigcap_{p \in V(I)} p = \sqrt{I}$$

Proposition.

closed sets of Spec A $\quad$ ideals

$$V(I) \longleftarrow I$$

$$Z \longrightarrow I(Z) := \bigcap_{p \in Z} p$$

$$\Rightarrow I(V(I)) = \sqrt{I}$$

This is similar to Hilbert Nullstellensatz!!

Definition. If $\sqrt{(0)} = (0)$,
we call $A$ reduced ring.

Clearly $A/I$ is reduced
$\Leftrightarrow$ $I$ is a radical ideal

Definition.
$X$ is a topology space.

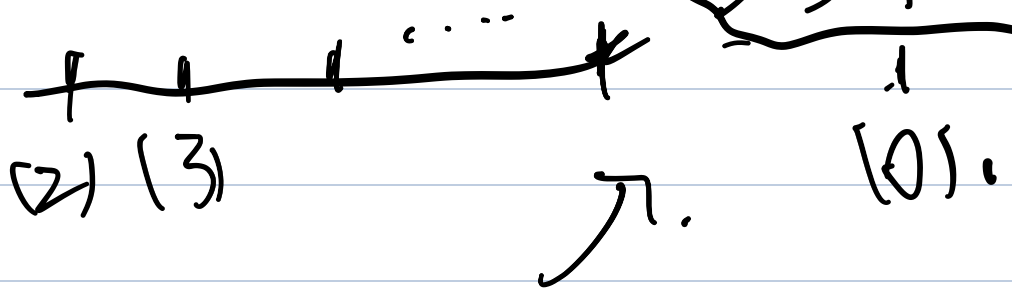
$X$ is irreducible
$\Leftrightarrow$ $X = X_1 \cup X_2$, $X_1, X_2$ closed
implies $X_1 = X$ or $X_2 = X$

$(\Leftarrow)$ every non-empty open set is dense.

E.g.

$\operatorname{Spec} \mathbb{Z} =$

$S \subseteq \operatorname{Spec} \mathbb{Z}$ closed $\Leftrightarrow$ $S = \operatorname{Spec} \mathbb{Z}$ or $S$ is finite subset of $\operatorname{Spec} \mathbb{Z} \setminus \{0\}$.




irreducible.

---

Proposition.

$$\begin{tikzcd} \{\text{irreducible closed subsets of } \operatorname{Spec} A\} \arrow[d, leftrightarrow, "1:1"] & V(\mathfrak{p}). \\ \operatorname{Spec} A & \mathfrak{p} \arrow[u, mapsto] \end{tikzcd}$$

Similar to algebraic varieties.

proof: Claim: $I$ is a radical ideal.

$V(I)$ is irre. $\Leftrightarrow$ $I$ is prime.

$\Leftarrow$: If $V(I) = V(I_1) \cup V(I_2)$

$V(I_1)$, $V(I_2)$ are both proper closed subsets.

$\Rightarrow I = \sqrt{I_1} \cap \sqrt{I_2}$

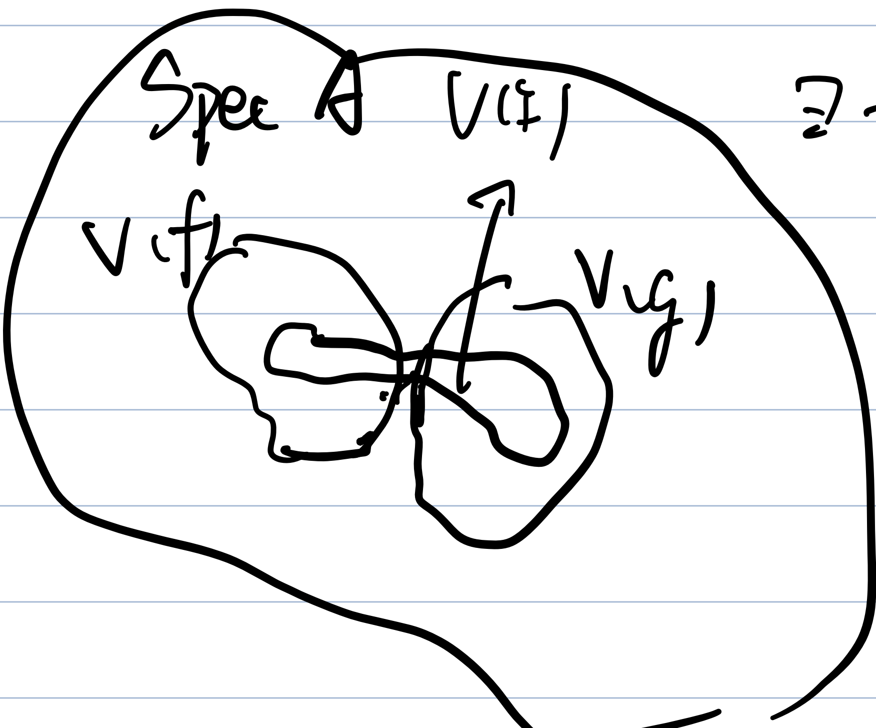
$\exists\, a \in \sqrt{I_1} \setminus I,\ b \in \sqrt{I_2} \setminus I$

$ab \in I$

Geometry view:

$V(I) \subseteq V(I_1) \cup V(I_2)$

$\exists\, f \in I_1,\ g \in I_2$




$V(f) \cup V(g) \supseteq V(I)$.

$f, g \notin I$

$I(V(fg)) \subseteq I(V(I))$

$\Rightarrow V(I) \subseteq V(fg)$

$\Rightarrow fg \in \sqrt{I} = I$

$\Rightarrow I$ is not prime

$\Rightarrow$ :

✓

D

Definition. Noetherian ring

$\Leftrightarrow$ every ideal is f.g.

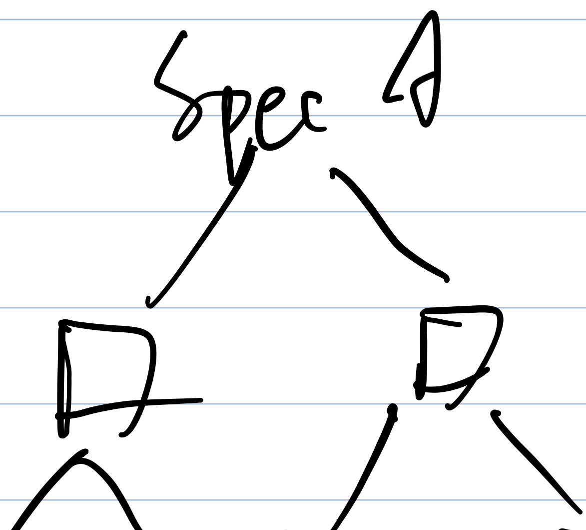
$\Leftrightarrow$ A.C.C

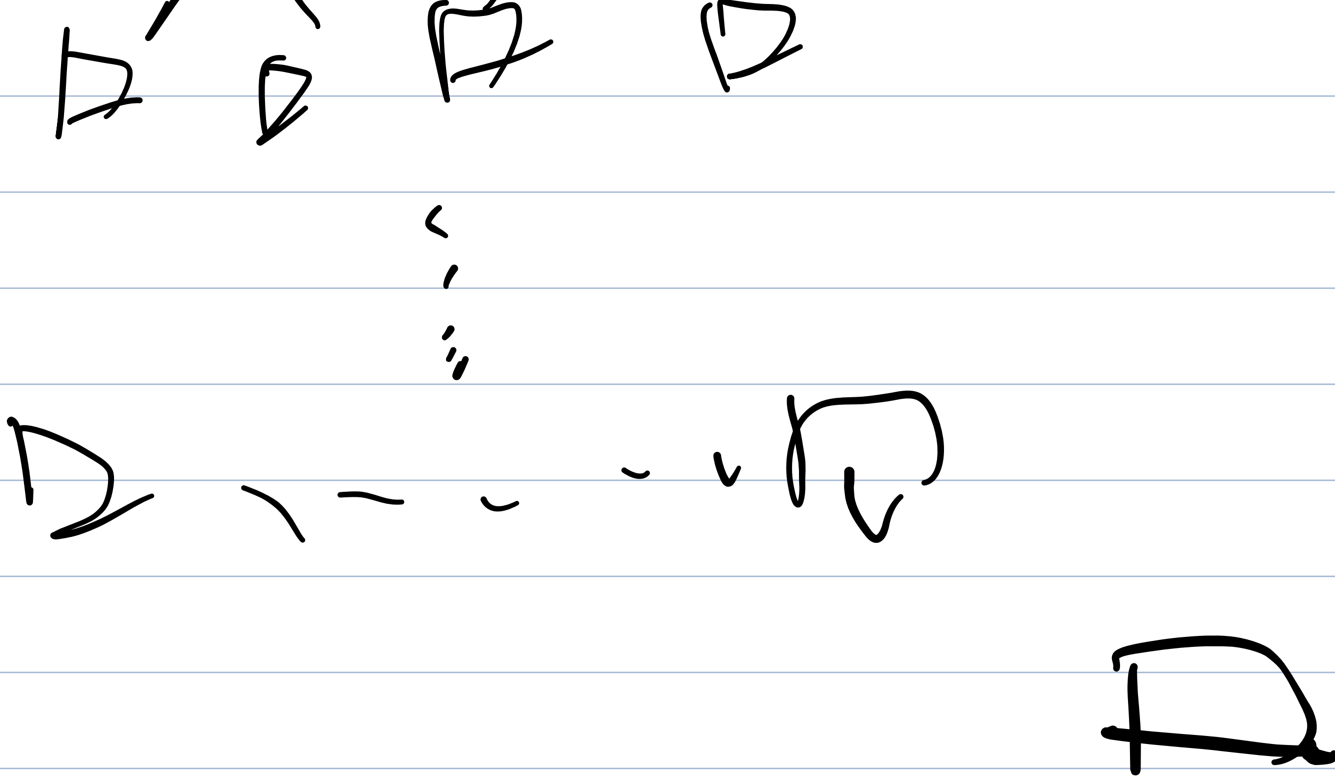
Proposition

A Noetherian

$\Rightarrow$ Spec $A$ is finite union of irre. closed subsets.

pf:





$X := \operatorname{Spec} A = X_1 \cup X_2 \cup \dots \cup X_n.$

$X_i$ irre., closed.

and $\forall i$, $X_i \not\subset \bigcup_{j \neq i} X_j$.

$\forall F$ irre, closed

$F = (F \cap X_1) \cup \dots \cup (F \cap X_n)$

$\Rightarrow \exists i,\ F = F \cap X_i$

Hence these $X_i$ are maximal irre. closed. subsets.

$$\{X_i\} \xleftrightarrow{1:1} \{\text{minimal prime ideals}\}$$

Corollary.

If $A$ is Noetherian

$\Rightarrow$ $A$ has only finite minimal primes.

Proposition.

$A \in$ comm rings.

$\Rightarrow$ every prime contains a minimal prime.

pf: Suppose $p$ prime

Let $\Sigma := \{ q \text{ prime} \mid q \subseteq p \}$.

By Zorn's lemma, $\Sigma$ has minimal element.

$\square$

---

Recall: $A$ comm ring

$$X = \operatorname{Spec} A.$$

- uniqueness
- glueing

proposition.

Suppose $\operatorname{Spec} A = D(f_1) \cup D(f_2)$

$g_1 \in A_{f_1}$ $\quad g_2 \in A_{f_2}$

$g_1 = g_2$ in $A_{f_1 f_2}$

$\Rightarrow \exists!\ g \in A$ s.t.

$$g = g_i \quad \text{in } A_{f_i}$$

$$g_1\big|_{D(f_1 f_2)} = g_2\big|_{D(f_1 f_2)}.$$

Proposition.

$D(f) \subseteq D(g)$

$\Rightarrow$ $\exists!$ homomorphism (Restriction map).

$$\rho_{D(g), D(f)} : A_g \to A_f$$

s.t.

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A_g \arrow[rr, "\rho"'] & & A_f \end{tikzcd}$$

$g \quad f_{D(g)D(f)}$

Lemma. $B$ ring.

$\forall p \in \operatorname{Spec} B$, $h(p) \neq 0$

$\Rightarrow h \in B^*$.

Pf: trivial.

Pf of proposition:

topology space.

Definition.

A (ab gps/ring/module) presheaf is

a functor from $X^{op}$ to

(ab gps/ring/module)

Definition. X topology space

B its basis.

B-presheaf is a functor

from $\mathcal{B}$ (subcategory of $X$) $\to$ (...).

Proposition. $X = \operatorname{Spec} A$, $\mathcal{B} = \{D(f) \mid f\}$.

$\forall\ u \in \mathcal{B}$, $u = D(f)$

$$\mathcal{O}_X(u) = A_f.$$

$D(f) = D(g)$

$\Leftrightarrow \quad \sqrt{(f)} = \sqrt{(g)}$

$\Rightarrow \exists x,\ f^n = gx$

$\exists y,\ g^m = fy.$

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] \arrow[ddl] \arrow[ddr] & \\
D(f) \arrow[rr] \arrow[d] & & D(g) \arrow[d] \\
D(f) & & D(g) \arrow[ll]
\end{tikzcd}$$

Definition.

$X$ topology space, $\mathcal{F}$ is presheaf.

Call $\mathcal{F}$ sheaf, If:

(1) $\forall\, U = \bigcup_i U_i$, $f|_{U_i} = g|_{U_i}$

$\Rightarrow f = g$

(2) $\forall\, U = \bigcup_i U_i$, $f_i \in \mathcal{F}(U_i)$

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

$\Rightarrow \exists f \in \mathcal{F}(U)$, $f|_{U_i} = f_i$

Definition: $\mathcal{B}$-sheaf.

Theorem.

$U = X = \operatorname{Spec} A$

$X = D(f_1) \cap D(f_2)$. $A$ integral domain

$\Rightarrow$ gluing

stalk.

$X$ topology space.

$\mathcal{F}(U) = \{\text{continuous } U \mapsto \mathbb{R}\}$

$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$.

Proposition.

$$A_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} := \varinjlim_{\mathfrak{p} \in D(f)} A_f$$

Proposition. $\mathcal{F}$ $\mathcal{B}$-presheaf

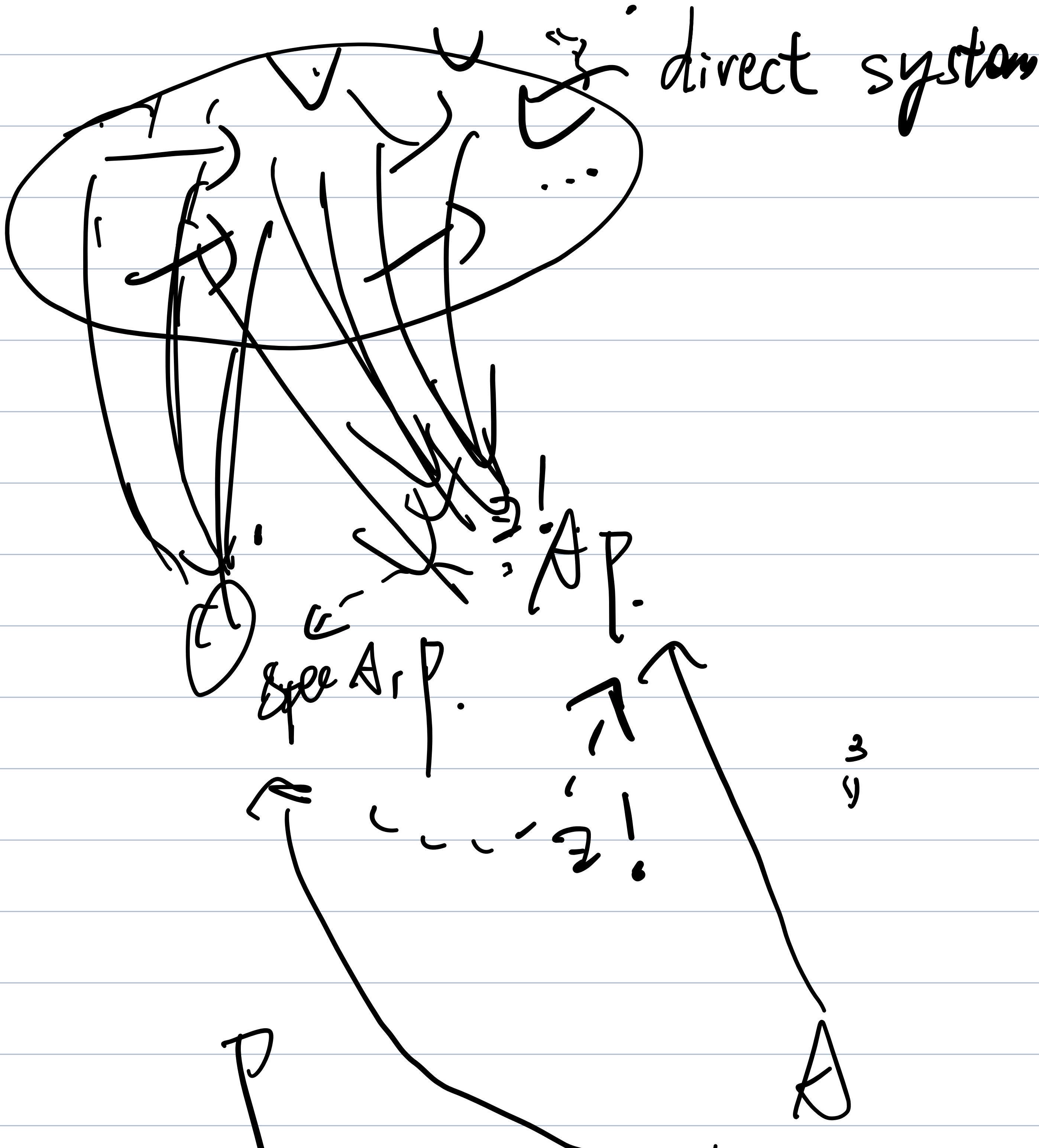
$S \subseteq \mathcal{B}$, s.t. $\forall U \in \mathcal{B}$, $x \in U$

$\exists V \in S$, $V \subseteq U$

$$\Rightarrow \varinjlim_{\substack{x \in U \\ U \in \mathcal{B}}} \mathcal{F}(\mathcal{B}) = \varinjlim_{\substack{x \in V \\ V \in S}} \mathcal{F}(V).$$

$$\mathcal{O}_{\operatorname{Spec} A, p} \xrightarrow{\sim} A_p$$

pf:



$\Downarrow$

$A_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{O}_{\operatorname{spec} A, \mathfrak{p}}$.

Proposition.

$\mathcal{F}$ is a $\mathcal{B}$-sheaf over $X$.

$s \in \mathcal{F}(U)$

$s_x = 0$, $\forall x \in U$

$\Rightarrow s = 0$.

pf: glueing.

Proposition.

Suppose $\operatorname{Spec} A$ is discrete.

$\Rightarrow \operatorname{Spec} A = \{P_1, \cdots, P_n\}$ and $P_i$ is maximal,

Pf: By compactness.

$\Rightarrow A \xrightarrow{\sim} A_{P_1} \times \cdots \times A_{P_n}$

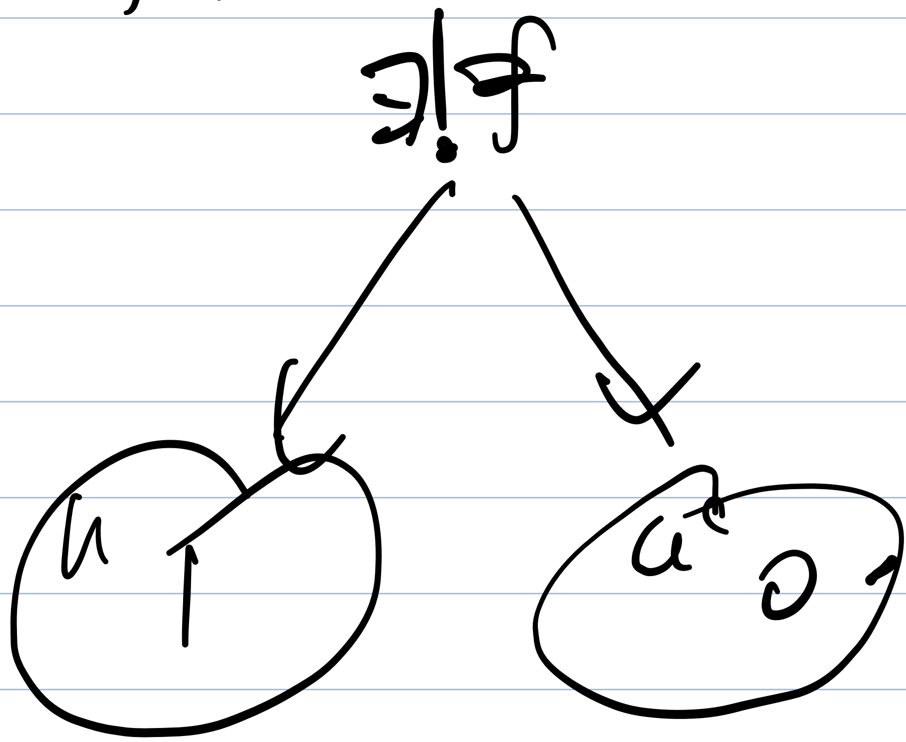
(glueing).

---

Proposition. $U \subseteq \operatorname{Spec} A$ open and closed

$\Rightarrow u = D(f)$

pf:




---

Proposition.

$$I = J \iff IA_p = JA_p, \ \forall p.$$

pf: $\Rightarrow$: trivial.

$\Leftarrow$: $f \in IA_p$

$\iff \bar{f} = 0$ in $A/I$

$\Leftrightarrow \bar{f} = 0$ in $(A/I)_P, \forall P$

$\Leftrightarrow \bar{f} = 0$ in $A_P / IA_P, \forall P$

$\Leftrightarrow$ in $A_P / JA_P$

$\Leftrightarrow \bar{f} = 0$ in $A/J$.

---

Module

$A$ ring $M$ is an $A$-module

Given $\varphi \in \mathrm{End}(M)$.

We can give $M$ an $A[x]$-module structure.

$f(x)\cdot v := f(x)(v)$

---

$$\begin{tikzcd} M \times N \arrow[d, "f_0 \text{ bilinear}"'] \arrow[drr, "f \text{ bilinear}"] & & \\ M \otimes N \arrow[rr, dashed, "\exists!\, g"'] & & K \end{tikzcd}$$

universal property of $M \otimes N$

$$M \otimes N := \frac{\text{free module generated by } M \times N}{\left\langle \begin{array}{l} (x, ay_1 + by_2) - a(x, y_1) - b(x, y_2) \\ (ax_1 + bx_2, y) - a(x_1, y) - b(x_2, y) \end{array} \right\rangle}$$

universal property of cokernel.

e.g.

$M = A[x]$ $N = A[y]$

$M \otimes_A N = A[x, y]$ (as A-algebra)

$M \times N \longrightarrow A[x, y]$

$(f(x), g(y)) \longmapsto f(x) g(y)$

induced

$M \otimes N \longrightarrow A[x, y]$

$f(x) \otimes g(y) \longmapsto f(x) g(y)$

$A[x, y] \longrightarrow M \otimes N$

$$x \longmapsto x$$
$$y \longmapsto y$$

# Property

$$M \otimes_A N = N \otimes_A M$$

$$\begin{tikzcd}
M \times N \arrow[r, shift left] \arrow[d] & N \times M \arrow[l, shift left] \arrow[d] \\
M \otimes N \arrow[r, dashed, shift left] & N \otimes M \arrow[l, dashed, shift left]
\end{tikzcd}$$

~~induced~~

$$M \otimes_A A \xrightarrow{\sim} M$$

$$\begin{tikzcd}
M \times A \arrow[d] \arrow[rd, "am"] & \\
M \otimes A \arrow[r, dashed] & M
\end{tikzcd}$$

$(m,a)$

$$M \otimes_A A/I \xrightarrow{\sim} M/IM$$

$$\begin{tikzcd}
M \times A \arrow[d] \arrow[rdd, bend left] & \\
M \times A/I \arrow[d] \arrow[rd, dashed] & \\
M \otimes A/I \arrow[r, dashed] & M/IM
\end{tikzcd}$$

$M \otimes_A (N_1 \oplus N_2) \xrightarrow{\sim} (M \otimes_A N_1) \oplus (M \otimes N_2)$

$\otimes_A$ is left adjoint

Hence preserve $\oplus$

$M \otimes_A A_S \xrightarrow{\sim} A_S$

---

Suppose $A \to B$ ring homomorphism

$M$ is $A$-module

$\Rightarrow$ $M \otimes_A B$ can naturally be viewed as $B$-module.

$b' \in B$

$$\begin{tikzcd} & M \times B \arrow[dl] \arrow[dr] & \\ M \otimes_A B \arrow[rr, dashed, "\exists !"'] & & M \otimes_A B \end{tikzcd}$$

$(x, b) \longmapsto x \otimes (bb')$

$x \otimes b \longmapsto x \otimes (bb')$

$M \otimes_A A_S \xrightarrow{\sim} M_S$ is both $A_S$, $A$ module.

Propositon.

$A^n \xrightarrow{\sim} A^m \Rightarrow n = m.$

Pf 1: $PQ = Id$ $P \in A^{n \times m}$, $Q \in A^{m \times n}$

$\Rightarrow n = m.$

pf 2: $p \in \operatorname{Spec} A$.

$$A \to \frac{A_p}{pA_p} = k(p)$$

$$A^n \otimes k(p) = \bigoplus_{i=1}^{n} \left(A \otimes \frac{A_p}{pA_p}\right)$$

$$= k(p)^n.$$

$A \to B \qquad A \to C$

Consider $B \otimes_A C$

Claim: $B \otimes_A C$ is $A$-algebra.

$$\left(\sum b_i \otimes c_i\right)\left(\sum b_j \otimes c_j\right) = \sum b_i b_j \otimes c_i c_j$$

is well-defined.

$$a \to \varphi(a) \otimes 1 = a \otimes 1$$

$$(ax) \otimes y = x \otimes (ay)$$

$\forall\, b_2 \in B \quad c_2 \in C$

$(b_1, c_1) \mapsto b_1 b_2 \otimes c_1 c_2$. bilinear map

$$\begin{tikzcd}
B \times C \arrow[d] \arrow[dr] & \\
B \otimes_A C \arrow[r, dashed, "\varphi_{b_2, c_2}"] & B \otimes_A C
\end{tikzcd}$$

$$b_1 \otimes c_1 \dashrightarrow b_1 b_2 \otimes c_1 c_2 .$$

Spec is a contravariant functor

from ComRings to Sets.

pullout fiber product.

$$\begin{tikzcd} B \otimes_A C & C \arrow[l, "i_2"'] \\ B \arrow[u, "i_1"] \arrow[ur, phantom, "\circlearrowright"] & A \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} B \otimes_A C \arrow[r, "i_2^*"] \arrow[d, "i_1^*"] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} C \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

Claim: $\operatorname{Spec} B \otimes_A C \overset{\text{surjective.}}{\twoheadrightarrow} \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C$

$$x \longmapsto (i_1^*(x),\ i_2^*(x))$$

$$\begin{tikzcd} S \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

$$S \times_X Y := \{(x, y) \in S \times Y \mid f(s) = g(y)\}$$

$$= \bigsqcup_{x \in X} \underbrace{f^{-1}(x)}_{\text{fiber}} \times \underbrace{g^{-1}(x)}_{\text{fiber.}}$$

fiber product in Sets.

Universal property:

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& S \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& S \arrow[r, "f"] & X
\end{tikzcd}$$

fiber product (pullback).

Specially, $k$ field

$\Rightarrow \operatorname{Spec}(B \otimes_k C) = \operatorname{Spec} B \times \operatorname{Spec} C$

$$\operatorname{Spec} k[x,y] \xleftrightarrow{1:1} k[x] \times k[y]$$

$$(f(x), g(y)) \longleftarrow (f(x)) \, ; \, (g(y))$$

$$\mathfrak{p} \in \operatorname{Spec} A$$

$$\downarrow$$

$$A \to k(\mathfrak{p}) = \frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}}$$

$$\mathfrak{p} \longleftarrow 0$$

pf : surjective :

$$\begin{tikzcd}
B \otimes_A C \arrow[d, dashed] & & C \arrow[ll] \arrow[dl] & \mathfrak{p}_2 . \\
k(\mathfrak{p}_1) \otimes_{k(\mathfrak{p})} k(\mathfrak{p}_2) / m & k(\mathfrak{p}_2) \arrow[l, "i_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & k(P_1) & k(P) \arrow[l] & \\
B \arrow[ur] & & & A \arrow[ul] \arrow[lll]
\end{tikzcd}$$

Not always injective: for $B = k[x]$, $C = k[y]$

$(x+y) \to (0,0)$

$(0) \to (0,0)$

If $\forall P_1 \in B$, $P = P_1 \cap A$

$A \to B$ induced

$k(P) \to k(P_1)$ is an iso.

$\Rightarrow$ It is a bijection.

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \frac{\mathbb{R}[x]}{(x^2+1)} = \frac{\mathbb{C}[x]}{(x^2+1)}$$

$$\overset{\text{CRT}}{\cong} \mathbb{C} \times \mathbb{C}$$

---

$k$ is a field. $V, W \neq 0$.

$\Rightarrow V \otimes_k W \neq 0$

Pf: $V \otimes_k W = (\bigoplus_i k) \otimes (\bigoplus_i W)$.

Cor. $\varphi: A \to B$ homomorphism.

$p \in \operatorname{Spec} A$.

$$\varphi^{*-1}(p) \overset{1:1}{\longleftrightarrow} \operatorname{Spec} B\otimes_A k(p)$$

$$\downarrow \wr$$

$$\operatorname{Spec} B_p/pB_p$$

$$\{0\}\times\varphi^{*-1}(p)$$

$$\|$$

Pf:
$$\begin{tikzcd}
\operatorname{Spec}(k(p)\otimes_A B) \arrow[r] \arrow[d] & \operatorname{Spec} B \supseteq \varphi^{*-1}(p) \arrow[d, "\varphi^*"] \\
\operatorname{Spec} k(p) \arrow[r] & \operatorname{Spec} A \\
0 & p
\end{tikzcd}$$

---

$\operatorname{Hom}_A(M,N)$. is $A$ module.

$M/N \quad M_S \quad M \otimes_A N \quad \mathrm{Hom}_A(M,N)$

$$\begin{tikzcd}
(M \otimes_A N)_S \arrow[rr, "\sim"] \arrow[dr, "\wr"'] & & M_S \otimes_A N_S \\
& & M_S \otimes_{A_S} N_S \arrow[u, "\wr"']
\end{tikzcd}$$

pf: $(M \otimes_A N)_S = (M \otimes_A N) \otimes_A A_S$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \downarrow\wr$

$M_S \otimes_A N_S = (M \otimes_A N) \otimes_A (A_S \otimes_A A_S)$

$(M_S \otimes_A A_S) \otimes_{A_S} (N \otimes_A A_S)$

$\parallel$

$$M_S \otimes_A (A_S \otimes_{A_S} N) \otimes_A A_S$$

$$\Big\downarrow \wr$$

$$M_S \otimes_A N_S .$$

---

$M$ A-module

$N$ B-module

$A \to B$ homomorphism.

$$\begin{tikzcd}
M \otimes_A N \arrow[r, "\sim"] \arrow[dr, "\wr"'] & (M \otimes_A B) \otimes_B N \\
& M \otimes_A (B \otimes_B N) \arrow[u, "\wr"']
\end{tikzcd}$$

Remark. Tensor product don't always preserve injective.

$$\mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto 2n \quad \text{injective}$$

But $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$

$$n \otimes 1 \mapsto 2n \otimes 1 = 0.$$

Adjoint relation.

$$\mathrm{Hom}_A(M_1 \otimes_A M_2, N) \xrightarrow{\sim} \mathrm{BiHom}(M_1 \times M_2, N)$$

$\cong Hom(M_1, Hom(M_2, N))$

That is, given a bilinear map.

Fix one coordinate, $\boxed{\text{obtain}}$ a homomorphism

$$Hom_A(\bigoplus_i M_i, N) = \prod_i Hom_A(M_i, N).$$

$$Hom_A(M, \prod_i N_i) = \prod_i Hom_A(M, N_i).$$

Exact sequence

$0 \to M \xrightarrow{\varphi} N \iff \varphi$ injective

$M \xrightarrow{\varphi} N \to 0 \iff \varphi$ surjective.

$0 \to M \xrightarrow{\varphi} N \to 0 \Leftrightarrow \varphi$ isomorphism.

$0 \to K \xrightarrow{i} M \xrightarrow{\varphi} N$.

$\Leftrightarrow$ $i$ injective.

$$\begin{tikzcd} K \arrow[r] \arrow[d] & M \\ \ker\varphi \arrow[ur] & \end{tikzcd}$$

$\exists!$. commutes

$\ker\varphi = i$

five lemma.

$M_1 \to M_2 \to M_3 \to 0$

right exact sequence

$\Rightarrow M_1 \otimes M \to M_2 \otimes M \to M_3 \otimes M \to 0$

Noetherian module:

Definition. call an A-module M

a Noetherian ring, if M satisfy

one of the following statement:

- $\forall N \subseteq M$, N is f.g.,
- Every ascending chain is stable

· Every set constituted by submodule of $M$ has a maximal element.

Proposition.

(1) $M$ Noetherian

$\Rightarrow M/N$ Noetherian.

(2) $M_S$ is a Noetherian $A_S$-module

(3) $M \otimes_A N$ ?

(4) $\mathrm{Hom}_A(M, N)$.

$A^m \to A^n \to M \to 0$

$0 \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(A^n, N) \to \mathrm{Hom}_A(A^m, N)$

$$\begin{matrix} \mathrm{Hom}_A(A^n, N) & & \mathrm{Hom}_A(A^m, N) \\ \| & & \| \\ N^n & & N^m \end{matrix}$$

$M$ is an $A$-module

$X = \mathrm{Spec}\, A$ $\qquad B = \{ D(f) \mid \forall f \in A \}$

$\tilde{M}(D(f)) = M_f$ $\qquad B$ - sheaf.

$D(f) = D(g) \Rightarrow M_f \xrightarrow{\sim} M_g$.

$$\tilde{M}_p = \varinjlim_{p \in D(f)} \tilde{M}(D(f)) \xrightarrow{\sim} M_p$$

Corollary.

$\operatorname{Spec} A = \{P_1, \cdots, P_n\}$ is finite,

every $P_i$ is maximal

$\Rightarrow M \xrightarrow{\sim} M_{P_1} \oplus M_{P_2} \oplus \cdots \oplus M_{P_n}$

$\mathcal{F}$ is a $\mathcal{B}$-sheaf

$\operatorname{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$

Proposition.

Let $A$ be a ring, $M$

a f.g. $A$-module

$\Rightarrow$ $\mathrm{Supp}(\tilde{M}) = V(\mathrm{ann}(M))$

$= \{a \in A \mid aM = 0\}$.

pf:

$M_p = 0 \iff \forall m \in M, \exists y \notin P$

$ym = 0$

$M$ is f.g.

$\iff$ $\exists y \notin P$, $yM = 0$

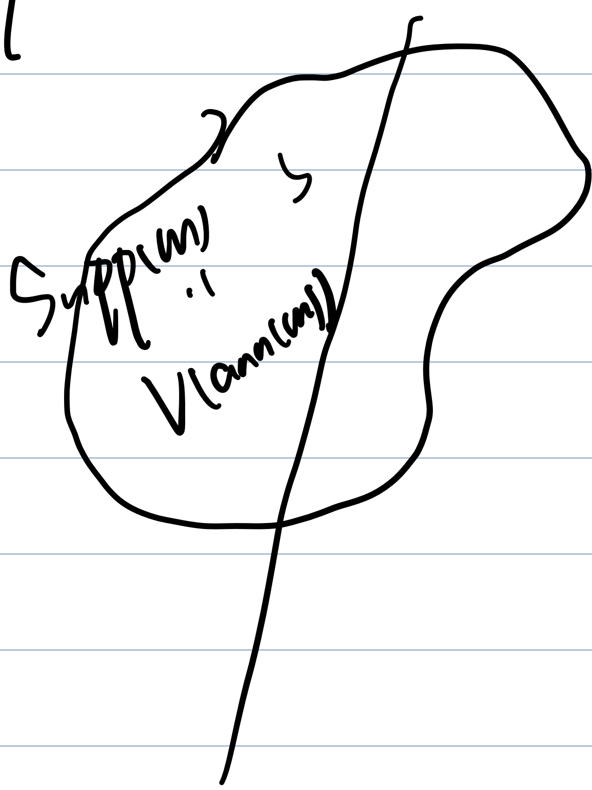
$\iff p \notin \mathrm{Ann}(M)$

$\square$

$\mathrm{Ann}(A/I) = I$

---

$M$ can be consider as a $A/\mathrm{ann}(M)$ module.



Determinant.

$\Rightarrow$ Nakayama's lemma

Nakayama's lemma

Suppose $(A, m)$ is a local ring

$M$ is a f.g $A$-module, and

$$mM = M$$

$$\Rightarrow \quad M = 0.$$

Or: $M$ is a f.g. $A$-module

$I \subseteq A$

$IM = M \Rightarrow \exists a \equiv 1 \bmod I$, s.t.

$aM = 0$.

---

Artin-Rees Lemma.

$A$ Noetherian. $M$ f.g $A$-module

$I \subseteq A$, $N \subseteq M$

$\Rightarrow \exists c$, s.t. $\forall i \geq c$

$I^i M \cap N = I^{i-c}(I^c M \cap N)$

Completion.

$I \subseteq A$

$$\hat{A} := \varprojlim A/I^n.$$

$$= \{(a_1, \ldots\ldots) \in \prod_{n=1}^{\infty} A/I^n \mid \pi_n(a_n) = a_{n-1}\}.$$

$$A/I^n \xrightarrow{\pi_n} A/I^{n-1}.$$

$\underleftarrow{\quad\quad}$

limit.

$s \in \hat{A}$ $\quad s = \sum_{i=0}^{+\infty} y_i, \quad y_i \in I^i.$

$t = \sum_{i=0}^{+\infty} z_i$

$$st = \sum_{i=0}^{+\infty} \sum_{k=0}^{i} y_k z_{i-k}.$$

Proposition.

$A$ is a Noetherian ring

$\Rightarrow \hat{A}$ is a Noetherian ring.

Proof. $I = (x_1, \cdots, x_n)$.

$\forall y_i \in I^i$

$$y = \sum_{u_1+\cdots+u_n=i} \square \, x_1^{u_1} \cdots x_n^{u_n}$$

$$\Rightarrow S = \sum_{i=0}^{+\infty} \sum_{u_1+\cdots+u_n=i} \square \, x_1^{a_1} \cdots x_n^{a_n}.$$

This form a surjection.

$$A[[x_1, \cdots, x_n]] \twoheadrightarrow \hat{A}.$$

↑.

ring of formal power series

Definition.

$M$ is an $A$-module, $I \subseteq A$.

$$\hat{M} = \varprojlim_{n} M/I^n M := \left\{ (x_1, x_2, \cdots) \in \prod_{n=1}^{\infty} M/I^n M \right\}$$

$\pi_n(x_n) = x_{n-1}$

is an $\hat{A}$ module.

$$A \xrightarrow{f} \hat{A}$$

$$a \to (a, a, \cdots)$$

If $A$ is a Noetherian local ring, $\hat{A}$ is the completion of $M$

$\Rightarrow A \to \hat{A}$ is injective.

$$\ker f = \bigcap_{a=1}^{\infty} I^i$$

Proposition.

$M$ is f.g $A$-module

$\Rightarrow \hat{M}$ is f.g $\hat{A}$-module

$$s = \sum_{i=0}^{\infty} y_i, \quad y_i \in I^i M$$

$$y_i = c_1^{(i)} x_1 + \dots + c_k^{(i)} x_k.$$

$$\Rightarrow s = \sum_{i=1}^{\infty} \sum_{j=1}^{k} c_j^{(i)} x_j$$

$$= \sum_{j=1}^{k} \left( \sum_{i=1}^{\infty} c_j^{(i)} \right) x_j.$$

is f.g. by $(x_1, \dots, x_n)$.

Corollary.

$f: M_1 \twoheadrightarrow M_2$ , $M_1, M_2$ f.g

$\Rightarrow \hat{M}_1 \twoheadrightarrow \hat{M}_2$ (check generators).

---

Proposition. $M_i$ f.g.

$0 \to M_1 \to M_2 \to M_3$ is exact

$\Rightarrow 0 \to \hat{M}_1 \to \hat{M}_2 \to \hat{M}_3$ is exact.

Proof:

$$\frac{M_3}{I^n M_3} = \frac{M_2/M_1}{I^n (M_2/M_1)} = \frac{M_2}{I^n M_2 + M_1}$$

$$= \frac{M_2/I^n M_2}{(I^n M_2 + M_1)/I^n M_2}$$

$$\frac{I^n M_2 + M_1}{I^n M_2} = \frac{M_1}{I^n M_2 \cap M_1}$$

$$0 \to \frac{M_1}{\underbrace{I^n M_2 \cap M_1}_{}} \longrightarrow \frac{M_2}{I^n M_2} \longrightarrow \frac{M_3}{I^n M_3}.$$

large. exact.

Apply Artin-Rees Lemma. $\exists\, c,\ \forall n > c.$

$$I^n M_2 \cap M_1 = I^{n-c} \underbrace{(I^c M_2 \cap M_1)}_{} \supseteq I^{n-c} M_1 \quad \therefore$$

$$\frac{M_1}{I^n M_1} \xrightarrow{\text{pr.}} \frac{M_1}{I^n M_2 \cap M_1} \qquad M_1$$

$$\begin{tikzcd}
\varprojlim \frac{M_1}{I^n M_1} \arrow[dr] \arrow[ddr, dashed, bend right] & & & \varprojlim \frac{M_1}{I^n M_2 \cap M_1} \arrow[dl] \arrow[ddll, dashed, bend left] \\
& \frac{M_1}{I^{n+1} M_1} \arrow[r, "p_{n-1}"] \arrow[d] & \frac{M_1}{I^{n+1} M_2 \cap M_1} \arrow[dl, "p"] & \\
& \frac{M_1}{I^{n+c} M_1} & &
\end{tikzcd}$$

This induced

$$\varprojlim M_1/I^n M_1 \xrightarrow{\sim} \varprojlim M_1/(I^n M_2 \cap M_1)$$

Apply. Left exactness of inverse limit.

$\square$

Proposition.

Suppose $A$ is a Noetherian ring, $M$ is a f.g $A$-module

$\Rightarrow \hat{M} = M \otimes_A \hat{A}$.

$$A^r \to A^s \to M \to 0$$

$$\begin{array}{ccccccc} \hat{A}^r & \to & \hat{A}^s & \longrightarrow & \hat{M} & \longrightarrow & 0 \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \end{array}$$

$$A^r \otimes_A \hat{A} \to A^s \otimes_A \hat{A} \to M \otimes \hat{A} \to 0.$$

Corollary.

$\otimes_A \hat{A}$ is exact.

( check f.g case is enough).

Corollary.

$J \subseteq A$, $A$ Noetherian.

$\hat{J}$ (at $I$).

$\Rightarrow$ $\hat{J} \hookrightarrow \hat{A}$

$\Rightarrow$ $\hat{J}$ is an ideal of $\hat{A}$ generated by $J$. $\hat{J} = J\hat{A}$

$J = (x_1, \cdots, x_n) \subseteq A$.

$\Rightarrow \hat{J} = (x_1, \cdots, x_n) \subseteq \hat{A}$

Proposition.

$(A, m, k)$ Noetherian. local ring.

$\Rightarrow (\hat{A}, \hat{m}, k)$ is a Noetherian local ring ( completion at $m \subseteq A$ ).

proof: (1) $\hat{A}/\hat{m} = \widehat{(A/m)} = k$

$$\begin{tikzcd} A/m \arrow[r, "\sim"] & k. \\ \dfrac{A/m}{m\cdot A/m} \arrow[u] \arrow[r, "\sim"] & k \arrow[u, "Id"'] \\ \vdots \arrow[u] \arrow[r, "\sim"] & k \end{tikzcd}$$

$m \cdot A/m = 0$

(2).

$s = y_0 + y_1 + y_2 + \cdots$

$y_k \in m^k$, $y_0 \notin m$.

$\Rightarrow \quad y_0^{-1} s = 1 + z_1 + \cdots\cdots$ is invertible

$(1+\varepsilon)^{-1} = 1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \dots$

$\Rightarrow s$ is invertible.

e.g.

$$\mathbb{C}[[t]] \xrightarrow{\sim} \varprojlim_{n} \mathbb{C}[t]/(t)^n.$$

$$\mathbb{C}[[t]] \longrightarrow \hat{\mathbb{C}[t]}$$

$$c_0 + c_1 t_1 + \cdots \longrightarrow (c_0 + c_1 t_1 + \cdots).$$

check this is inj. surj.

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/(p^n)$$

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ \mathbb{Z}/p^2 & & \square \\ \downarrow & & \downarrow \\ \mathbb{Z}/p & & \square \end{array}$$

$$c_0 + c_1 p + c_2 p^2 + \cdots \qquad c_i \in \{0, \dots, p-1\}.$$

this express is unique.

$$(p-1)p + (p-1)p^2 + \cdots$$

$$= p(p-1)\cdot(1+p+\cdots)$$

$= p \cdot (p-1) \cdot \frac{1}{1-p} = -p.$

$\mathbb{Z} \hookrightarrow \mathbb{Z}_p.$

$(\mathbb{Z}_p, p\mathbb{Z}_p, \mathbb{F}_p)$ is a Noetherian

Local ring.

$(\mathbb{Z}_{(p)}, p\mathbb{Z}_{(p)}, \mathbb{F}_p)$ ($\mathbb{Z}_{(p)}$ is $\underline{\text{Localization}}$?)

$$\frac{\mathbb{Z}_{(p)}}{p^n \mathbb{Z}_{(p)}} = \left(\frac{\mathbb{Z}}{(p^n)}\right)_{(p)} = \frac{\mathbb{Z}}{(p^n)}.$$

$\hat{\mathbb{Z}}_{(p)} = \mathbb{Z}_p.$

Recall, $f \in A$

$$f = 0 \in A_p \Leftrightarrow f = 0,$$

$$A \hookrightarrow \prod_{p \in \operatorname{Spec} A} A_p$$

$\operatorname{Ass}(A)$ 伴随素理想.
(associated prime ideal.)

$$A \hookrightarrow \prod_{p \in \operatorname{Ass}(A)} A_p$$

Definition.

If $p \in \operatorname{Spec} A$, $\exists x \in A$

$$p = \operatorname{ann}(x) = \{a \mid ax = 0\};$$

call $p$ an associated prime ideal.

$\boxed{\underline{\operatorname{Ass}(A)}}$.

Example.

$$\operatorname{Ass}(\mathbb{Z}) = \{(0)\}.$$

Definition.

Suppose $M$ is an $A$-module,

$\mathfrak{p} \in \operatorname{Spec} A$.

$\mathfrak{p}$ is called an associated prime

of $M$, if $\exists x \in M$, s.t.

$$\mathfrak{p} = \operatorname{ann}(x) = \{a \mid ax = 0 \in M\}.$$

$\operatorname{Ass}(M)$ or $\operatorname{Ass}_A(M)$.

- $\mathfrak{p} \in \operatorname{Ass}(M) \Longleftrightarrow_{\exists \text{ inj},} A/\mathfrak{p} \hookrightarrow M$

$\Rightarrow$: $\mathfrak{p} = A/\mathfrak{p}$.

$a \rightarrow ax$.

$\Leftarrow$ : Let $a = f(1)$.

$\Rightarrow$ $\mathfrak{p} = \mathrm{ann}(1)$.

Proposition.

Let

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

be an $A$-module exact sequence.

$\Rightarrow$ $\mathrm{Ass}(M_1) \subseteq \mathrm{Ass}(M_2) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_3)$

pf:

$\mathrm{Ass}(M_1) \subseteq \mathrm{Ass}(M_2)$ :

$A/p \hookrightarrow M_1 \hookrightarrow M_2$ ✓.

$Ass(M_2) \subseteq Ass(M_1) \cup Ass(M_3)$:

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

$$\begin{array}{c} M_2 \\ \uparrow i \\ A/p \end{array}$$

$$i(A/p) = N \subseteq M_2$$
$$\begin{array}{c} N \\ \wr\!\wr \\ Ax \end{array}$$

Case 1: $N \cap M_1 \neq 0$.

$$\Downarrow$$
$$a \neq 0$$

$\Rightarrow Ann(a) = p$

Case 2: $N \cap M_1 = 0$

## Integral extension.

**Definition.** $A \to B$

$b \in B$ is integral over $A$ if $\exists a_i \in A$

$$b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0$$

$\forall b \in B$ $b$ integral over $A$

$\Rightarrow$ $B$ is integral over $A$.

**Proposition.** $A \to B$ $b \in B$., TFSAE:

i) $b$ is integral over $A$.

(2) $A[b]$ is f.g. $A$-mod.

(3) $\exists\, C \subseteq B$ s.t. $C$ is a f.g. $A$-mod.
$\underset{b\in}{C}$

$$A \subseteq \underset{\substack{\Downarrow \\ b}}{C} \subseteq B$$

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clearly.

(3) $\Rightarrow$ (1) ( Caley . Hamilton).

$C = Ax_1 + \cdots + Ax_n$.

$\Rightarrow bC \leq C$

$$\varphi_b : \begin{aligned} C &\to C \\ c &\to bc \end{aligned}$$

$\Rightarrow \varphi_b^n + P\varphi_b^{n-1} + \cdots + D = 0$

$\Rightarrow b^n + \square b^{n-1} + \dots + \square = 0$.

$\square$

Corollary. $A \to B$, $b_1, b_2$ integral

$\Rightarrow b_1 + b_2,\ b_1 \cdot b_2$ is integral.

$$\begin{array}{c} A[b_1, b_2] \\ \cup| \\ A[b_1] \\ \cup| \\ A \end{array}$$

Definition. $A \to B$. $\{ b \in B \mid b \text{ is integral} \}$

is called by integral closure of $A$ in

$B$.

ring of integers.

$$\begin{tikzcd} \mathcal{O}_K \arrow[r, phantom, "\subseteq"] & K \\ \mathbb{Z} \arrow[u] \arrow[r, hook] & \mathbb{Q} \arrow[u, "\text{finite extension.}"'] \end{tikzcd}$$

Example.

$$\begin{tikzcd} \mathbb{Z}[i] \arrow[r, hook] & \mathbb{Q}(i) \\ \mathbb{Z} \arrow[u, hook] \arrow[r, hook] & \mathbb{Q} \arrow[u, hook] \end{tikzcd}$$

Proposition.

$A \longrightarrow B \qquad B = A[b_1, \dots, b_n]$

Then $B$ is integral over $A$

$\Leftrightarrow$ $b_i$ is ~

$\Leftrightarrow$ $B$ is f.g. $A$-mod.

---

$A \to B$ integral.

$$\begin{tikzcd} C\otimes_A B & B \arrow[l] \\ C \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

$\Rightarrow$ $C \to C\otimes_A B$ is integral.

(basis change).

$(1\otimes b)^n + 1\otimes(a_{n-1}b)^{n-1} + \dots = 0$

Cor. $B/I$ is integral over $A$;

$B_S \sim A_S$

Proposition. $A, B$

$A \hookrightarrow B$, $A \to B$ is integral

$\Rightarrow$ $A$ is a field $\Leftrightarrow$ $B$ ~

proof:

$\Rightarrow$ $0 \neq b \in B$

$(b^{n-1} + D\, b^{n-2} + \dots + D)\, b + a_0 = 0.$

$\Leftarrow$: $a \in A$ $a^{-1} \in B$

$(a^{-1})^n + \square (a^{-1})^{n-1} + \dots + \square = 0.$

$\Rightarrow$ $a^{-1} + \square + \square a + \dots + \square a^{n-1} = 0$

$\Rightarrow a^{-1} \in A$.

Cor. $A \hookrightarrow B$ integral

$m \subseteq B$ maximal

$\Leftrightarrow m \cap A \subseteq A$ maximal.

Pf: $A/m \cap A \longrightarrow B/m$.

Cor. $A \xrightarrow{\varphi} B$ integral.

$\mathfrak{q}_1, \mathfrak{q}_2 \in \varphi^{-1}(p)$

$\Rightarrow \mathfrak{q}_1 \not\subseteq \mathfrak{q}_2, \ \mathfrak{q}_2 \not\subseteq \mathfrak{q}_1.$

Pf:

$$\begin{tikzcd} \overset{\text{maximal}}{B_p} & B \arrow[l] \\ A_p \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

$p \in \operatorname{Spec} A.$

$pA_p.$

$\Rightarrow$ Every element in $\varphi^{-1}(p)$ is maxmal, hence is closed.

Cor. $A \xrightarrow{\varphi} B$ integral

$\Rightarrow \forall J \subseteq B$, $\varphi^*(V(J)) = V(J \cap A)$

$\Leftarrow$ ?

$\because$

$$\begin{tikzcd}
A/J\cap A \arrow[r, hook] & B/J. \\
A \arrow[u] \arrow[r] & B \arrow[u]
\end{tikzcd}$$

$\varphi^*(V(J)) = \{p \cap A \mid p \in V(J)\} \subseteq V(J \cap A)$

$A \hookrightarrow B$

$\Rightarrow$ $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$. surj ?

$p \in A$.

$\Rightarrow$, $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{P}}$, $\mathfrak{P}$. maximal.

$\square$.

例. $\mathbb{Z}[i]$ $\qquad$ $(0) \neq \mathfrak{p} \subseteq \mathbb{Z}[i]$

$$\begin{array}{ccc} \mathbb{Z}[i] & & \mathfrak{p} \\ \uparrow & & \downarrow \\ \mathbb{Z} & & \mathfrak{p} \cap \mathbb{Z} \end{array}$$

If $\mathfrak{p} \cap \mathbb{Z} = (0)$

$(0) \cap \mathbb{Z} = (0)$

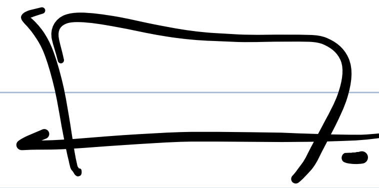
$\Rightarrow (0) \subseteq \mathfrak{p}$ $\qquad$ $\times$

$\Rightarrow \quad p \cap \mathbb{Z} \neq (0)$

Exercise.

$A \hookrightarrow B$ inj.

$\Rightarrow \varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is dominant. i.e. $\overline{\operatorname{Im} \varphi^*} = \operatorname{Spec} A$.

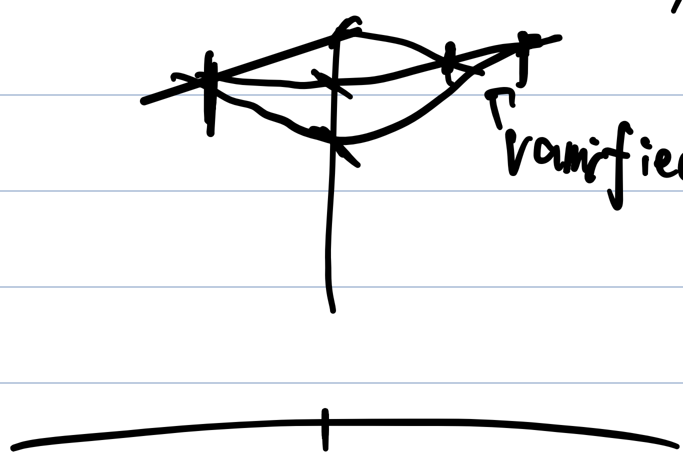


$$\mathbb{C}[X] \hookrightarrow \frac{\mathbb{C}[X,Y]}{(y^3 - xy^2 + 1)}$$

$(x-a)$ $\qquad (x-a, y-b)$.

$a^3 - ab^2 + 1 = 0$.

$(x-a, y-b)$, $a^3 - ab^2 + 1 = 0$.



ramified points

$(x-a)$

$\mathbb{C}$

Definition.

$A \to B$ finite extension

$\Leftrightarrow$ $B$ is f.g. $A$-mod.

finite $\Rightarrow$ integral.

Proposition.

$A \to B$ finite

$\Rightarrow \operatorname{Spec} B \to \operatorname{Spec} A$ is closed, &

$\boxed{\text{Every fiber is finite}}$

$$\begin{tikzcd} B_p/pB_p & B \arrow[l] \\ k(p) \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

$A \supseteq p$

$B_p/pB_p \cong k(p) \otimes B$ is f.g. $k(p)$ extension.

Every elements of $\operatorname{Spec} B_p/pB_p$

is minimal.

Theorem. Noether Normalization

Lemma.

Let $k$ be a field, $A = \frac{k[x_1, \cdots, x_n]}{I}$

f.g. $k$-algebra.

$\Rightarrow \exists t_1 \sim t_m \in A$, $t_1 \sim t_m$ algebraic

independent, $k[t_1, \cdots, t_m] \hookrightarrow A$ f.g.

extension. ( i.e. $A$ is f.g. $k[t_1, \cdots, t_m]$

module)

$m$ : transcendental degree.

☆ Theorem.

$A$ is a f.g. algebra over $k$, $p \in \operatorname{Spec} A$, then $p$ is maximal

$\Leftrightarrow [K(p):k] < +\infty$ $\quad K(p) \hookrightarrow K(p) \otimes_k \bar{k}$

$\mathfrak{a} \cap K(p) = 0.$

pf:

$\Rightarrow$ : $\dfrac{k(p) \otimes_k \bar{k}}{\mathfrak{a}}$

$\bar{k}$ $\qquad\qquad$ $k(\mathfrak{p})$.

$\Leftarrow$: $k \to A/\mathfrak{p}$ finite extension

$\Rightarrow A/\mathfrak{p}$ is a field.

$\square$

Corollary.

$A/k$ f.g. algebra.

$\mathfrak{p} \in \operatorname{Spec} A$. $\mathfrak{p} \in D(f) = \operatorname{Spec} A_f$

$\Rightarrow$ $\mathfrak{p} \in \operatorname{Spec} A$ is a closed point

$\Leftrightarrow$ $\forall \mathfrak{p} \in D(f)$

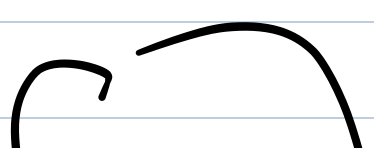
$\{\mathfrak{p}\}$ is closed in $D(f)$.

$(k(\mathfrak{p}) = k(\mathfrak{p}A_f))$

Cor.

$A/k$ f.g. algebra.

$\Rightarrow$ $\{\mathfrak{p} \in \operatorname{Spec} A \mid \{\mathfrak{p}\} \text{ is closed}\}$ is dense




maximal.

Theorem. (Hilbert's Nullstellensatz).

$I(V(J)) = \sqrt{J}$, $k$ algebraic closed.

Pf: $g \in I(V(J))$

$g(a) = 0$, $\forall a \in V(J)$

$$A = \frac{k[x_1, \dots, x_n]}{J}$$

$\Rightarrow \operatorname{Spec}_m A = V(J)$

$\Rightarrow A_g \quad \operatorname{Spec}_m A_g = \operatorname{Spec}_m A \cap D(g)$

$= \phi$

$\Rightarrow A_g = 0 \Rightarrow g \in \sqrt{J}.$

$\square$

Cor. $A/k$ f.g. algebra

$\Rightarrow \sqrt{I} = \bigcap\limits_{\substack{m \supseteq I \\ m \text{ maximal}}} m$

Pf: Suppose $I = 0$.

$\forall f \in \bigcap \mathfrak{m}$

$\mathfrak{m}$ maximal.

$$\operatorname{Spec}_m A_f = \operatorname{Spec}_m A \cap D(f) = \emptyset.$$

Dedekind's domain.

1.
$$\begin{array}{ccc} \mathcal{O}_K & \subseteq & K \\ \uparrow & & \uparrow \\ \mathbb{Z} & \subseteq & \mathbb{Q} \end{array}$$

$\mathcal{O}_K$ is a Dedekind's domain.

2.

$$
\begin{tikzcd}
\mathrm{Frac}\left(\mathbb{C}[X,Y]/(X^3+Y^3+1)\right) \arrow[r, phantom, "\supseteq"] \arrow[d, dash] & \dfrac{\mathbb{C}[X,Y]}{(X^2+Y^3+1)} \arrow[d, dash] \\
\mathbb{C}(X) \arrow[r, phantom, "\supseteq"] & \mathbb{C}[X].
\end{tikzcd}
$$

Discrete valuation ring. DVR.

Definition: $(A, m)$ Noetherian local ring, integral, $m = (\pi)$ is principle

$\Longleftrightarrow$ $A$ is a DVR

$\mathbb{C}[[X]]$ $\quad m = (X)$.

$\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid (b,p)=1\}$ $\quad m = p\mathbb{Z}_{(p)}$.

localization

$f \in I$ $\quad f = a_k X^k + \square X^{k+1} \cdots\cdots$

invertible

$= X^k \cdot (a_k + \cdots)$

Proposition.

$(A, m)$ DVR

$m = (\pi)$

$\Rightarrow \forall a \in A,\ a \neq 0$

$\exists!\, k$, s.t. $\exists u \in A^*$

$a = u \cdot \pi^k$ $\quad a \in m^k / m^{k+1}$

proof: $\bigcap_{k=1}^{\infty} m^k = (0)$

$\Rightarrow \exists k,\ a \in m^k / m^{k+1}$

Corollary.

Every non-zero ideal $I \subseteq A$ can be expressed as $m^k$

$\Rightarrow$ DVRs $\subseteq$ PIDs $\subseteq$ UFDs

Def

$A$ is a integral closed domain

if $A$ is an integral domain, and

the integral closure of $A$ in $\mathrm{Frac}(A)$

is $A$.

Proposition. UFD is integral closed

$x = \frac{a}{b}$, $a, b \in \mathrm{Frac}(A)$

$\gcd(a,b) = 1$

$x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0 \quad c_i \in A$

$$a^n + b \cdot \boxed{\ ?\ } = 0 \quad \times.$$

$$\uparrow$$
$$A.$$

Proposition.

$A$ is integral, then $A$ is integral closed $\Leftrightarrow$ $\forall \mathfrak{p} \in \operatorname{Spec} A$, $A_{\mathfrak{p}}$ is integral closed

Pf: $K = \operatorname{Frac}(A)$

$A_{\mathfrak{p}} \subseteq K$, $\bigcap_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}} = A$ $\quad \begin{matrix} K \\ \cup \\ A_{\mathfrak{p}} \end{matrix}$

"$\Leftarrow$": $x \in K$.

$x$ integral over $A$

$\Rightarrow$ $x$ integral over $A_p$

$\Rightarrow$ $x \in A_p \Rightarrow x \in \bigcap_p A_p = A$

"$\Rightarrow$":

$$x^n + \frac{a_{n-1}}{s_{n-1}} x^{n-1} + \cdots + \frac{a_0}{s_0} = 0$$

$s_i \notin P$

$\Rightarrow s_{n-1} s_{n-2} \cdots s_0 x \in A$

$\Rightarrow \quad x \in A_p.$

Definition.

Krull demension of $A$

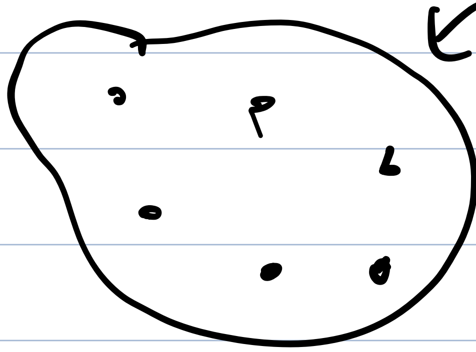
$= \sup \{ r \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r,\ P_i \in \operatorname{Spec} A \}$

$\dim A = 0 \Leftrightarrow \operatorname{Spec} A = \operatorname{Spec}_m A.$

$\mathbb{C}[x] / (x^2)$

Recall. $A$ Noetherian, $\dim A = 0$

$\Rightarrow A \xrightarrow{\sim} A_{p_1} \times \cdots \times A_{p_m}$

Spec A  discrete space.

glueing.

$A \in$ DVRs $\Rightarrow \dim A = 1$

Proposition. A is a Noetherian.

Local. integral, $\dim A = 1$

$\Downarrow$

$A$ is a DVR.

$(A, m)$ $\quad a \in m \quad a \neq 0.$

$$\frac{m}{(a)} \subseteq \frac{A}{(a)} \quad \text{minimal prime}$$

$$a^n + c_{n-1} a^{n-1} + \cdots + c_0 = 0.$$

$$\Rightarrow \frac{m}{(a)} = \mathrm{Ann}(\bar{b})$$

$$m \cdot \frac{b}{a} \subseteq A.$$

(1) $m \cdot \frac{b}{a} \subseteq m.$

$$\Rightarrow \quad m = A x_1 + \dots + A x_n .$$

$$\frac{b}{a} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} .$$

$$\Rightarrow \det\left( \frac{b}{a} I_n - \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \right) = 0$$

$\Rightarrow \frac{b}{a}$ integral over $A$.

$$\Rightarrow \frac{b}{a} \in A . \qquad \times .$$

$$\Rightarrow \quad m \cdot \frac{b}{a} = A .$$

$$\Rightarrow \exists \pi \in m , \quad \pi \cdot \frac{b}{a} = 1$$

$\Rightarrow \mathfrak{m} = \mathfrak{m}\cdot\pi\cdot\frac{b}{a} = A\pi$

$\square$

Definition.

$\mathfrak{p} \in \operatorname{Spec} A$.

$\operatorname{ht}\mathfrak{p} := \dim A_{\mathfrak{p}}$

$= \sup\{ r \mid \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \}$.

ht p.

$\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p} \subseteq \mathfrak{p}_{k+1} \subseteq \cdots\cdots$

$\operatorname{ht}\mathfrak{p}$ 为 $\mathfrak{p}$ 在链中位置.

(从左开始数).

- $\mathfrak{p}$ minimal $\Leftrightarrow$ $\mathrm{ht}\,\mathfrak{p}=0$.

- $\mathrm{ht}\,\mathfrak{p}=1 \Leftrightarrow$. $\begin{array}{c}\mathfrak{p}\\ \rotatebox{90}{$\subsetneq$}\\ \mathfrak{p}_0\end{array}$ 之 X.

Proposition. $A$ Noetherian, integral, integral closed.

$\mathfrak{p}\in \mathrm{Spec}\,A$, $\mathrm{ht}\,\mathfrak{p}=1$

$\Rightarrow A_p$ is a DVR

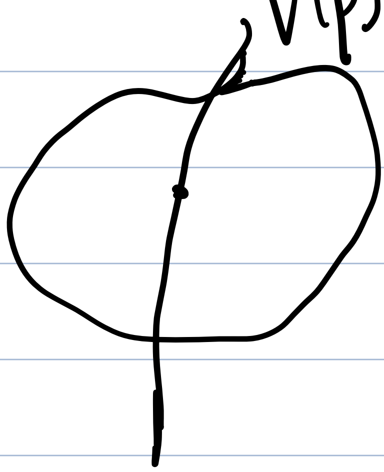
$\square$

Proposition.

$A$ Noetherian, integral, integral closed

$\Rightarrow A = \bigcap_{\mathrm{ht}\, p = 1} A_p$.

$V(p)$: irre. curve. hypersurface.



$\mathbb{C}^2$.

pf:

It will be enough to pr

$\bigcap_{ht\,p=1} A_p \subseteq A.$

$x \in \bigcap_{ht\,p=1} A_p$

$I := \{ b \in A \mid bx \in A \}.$

① $I = A \Rightarrow x \in A.$

② $I \subsetneq A$

$x = \frac{a}{c}$, $c \neq 0$, $a \notin (c)$

$I = \{ b \in A \mid ba \subseteq (c) \}$.

$$\begin{array}{ccc} I/(c) = & \operatorname{ann}(a) & \subseteq p \\ & \cup & \\ & A/(c) & \end{array}$$

(as a prime ideal of $A/(c)$)

Let $p \in \operatorname{Ass}(A/(c))$

$p = \operatorname{ann}_A(\bar{y})$

$p \in \operatorname{Ass}((A/(c))_p)$

$p \cdot y \subseteq (c)$

$$p \cdot \frac{y}{c} \subseteq A_p.$$

(1)

$$p \cdot \frac{y}{c} = A_p.$$

$$\pi \cdot \frac{y}{c} = 1$$

$$\Downarrow$$

$$p = (\pi)$$

$$\Rightarrow A_p \in \text{DVRs}$$

$$\Rightarrow \operatorname{ht} p = 1$$

(2) $$p \cdot \frac{y}{c} = p A_p$$

Recall. $(A, m)$ local. Noetherian.

integral. domain $m = (\pi)$

$\Rightarrow$ DVR.

Definition.

$A$ integral domain.

$K = \mathrm{Frac}(A)$.

$A^{\gamma} \subseteq K$ is integral closure of $A$.

$$A = A^{\nu} \Rightarrow A \text{ regular}$$

(integral closed).

DVR $\Rightarrow$ regular $\Leftarrow$ UFD

Proposition.

$A$ is a DVR

$\Leftrightarrow$ $A$ is a Noetherian, regular, local, integral domain, dimension 1.

# Dedekind domain.

Def.

$A$ integral, Noetherian.

$\forall\, \mathfrak{p} \in \operatorname{Spec} A$, $A_{\mathfrak{p}}$ is a DVR

then call $A$ a Dedekind domain.

($\Leftrightarrow$ $\forall\, \mathfrak{m} \in \operatorname{Specm} A$, $A_{\mathfrak{m}}$ is a DVR)

Proposition.

$A$ is a Dedekind domain

$\Leftrightarrow$ $A$ is a dimensional 1, regular

Noetherian domain.

Pf: $\Rightarrow$ : trivial.

$\Leftarrow$ : trivial.

PID $\Rightarrow$ Dedekind domain.

---

Example.

$k = \bar{k}$

$A = k[x,y]/(f(x,y))$

$f(x,y) \in k[x,y]$ irreducible.

If $V(f)$ is smooth

$\Rightarrow A$ is regular

Pf: $m = (\overline{x-a}, \overline{y-b}) \subseteq A$

$a = b = 0 \qquad f(0,0) = 0.$

$\Rightarrow f_x(0,0) \neq 0$

$f(x,y) = x + c \cdot y + \underbrace{g(x,y)}_{\text{sum of terms with degree} \geq 2.}$.

$(A_m, mA_m = (\bar{x}, \bar{y}))$ $\nearrow mA_m$.

$x \equiv -cy \mod m^2$ in $A_m$.

$f(\bar{x}, \bar{y}) = 0$

$\bar{x}(\underbrace{1 + g_1(\bar{x}, \bar{y})}) = -c\bar{y} + g_2(\bar{y})$.

$\uparrow$ $1 + m$ invertible

$\uparrow$ $(y)$.

$\Rightarrow \bar{x} \in (\bar{y})$

$\Rightarrow mA_m = (\bar{y})$

$\Rightarrow A_m$ is a DVR.

---

$A \hookrightarrow B$ map of local ring

$mB = n^e$

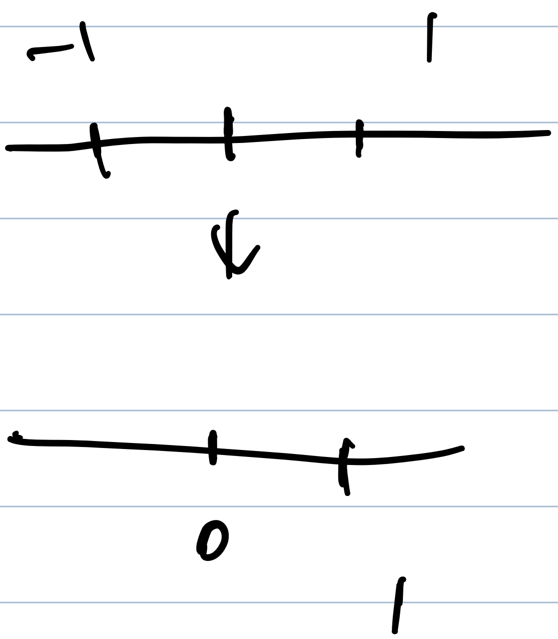
Definition.

$e$: ramified index

(分歧指数)

$$k[y] \qquad\qquad y_0$$
$$\uparrow \qquad\qquad\quad |$$

$\mathbb{k}[x]$ $\qquad$ $y_0^2$




$(x) \to (y^2)$

---

$\mathbb{R}[x] \to \mathbb{R}[y]$

$x \mapsto y^2$

$m = (x+1)$. $\qquad$ $n = (y^2+1)$.

$(\mathbb{R}[x]_m, m) \longrightarrow (\mathbb{R}[y]_n, n)$

$$\begin{array}{cc} \parallel & \parallel \\ (x+1) & (y^2+1). \end{array}$$

$e = 1$

Theorem. $A$ Dedekind domain

$K = \mathrm{Frac}(A)$

$L/K$ finite separable field extension.

$B \subseteq L$ integral closure

- $B/A$ finite extension.

Then:

① $B$ is a Dedekind domain.

② $\{\mathfrak{O} \in \operatorname{Spec} B \mid \mathfrak{O} \cap A = P\}$

is finite. $\mathfrak{O}_1 \sim \mathfrak{O}_t$

③ $A_P \to B_{\mathfrak{O}_i}$ $e_i$, $f_i$

$(f_i = [B_{\mathfrak{O}_i}/\mathfrak{O}_i B_{\mathfrak{O}_i} : A_P/PA_P])$

$$\sum_{i=1}^{t} e_i f_i = [L:K]$$

$$= [B:A]$$

④ $PB = \mathfrak{Q}_1^{e_1} \cdots \mathfrak{Q}_t^{e_t}$

$\Downarrow$

$B$ as free $A$-module.

Pf:

**定理 4.3.1** 设 $A$ 为 Dedekind 整环，$K = \mathrm{Frac}(A)$ 为其分式域．设 $K \hookrightarrow L$ 为域的有限扩张．令 $B := \{x \in L \mid x \text{ 在} A \text{ 上整}\}$ 为 $A$ 在 $L$ 中的整闭包．设 $B$ 为有限 $A$-模，$P$ 为 $A$ 的非零素理想，则有:

(i) $B$ 为 Dedekind 整环．

固定P.

非平凡.

A=Z时对.





(ii) $B$ 中满足 $Q \cap A = P$ 的素理想 $Q$ 均为非零素理想，并且只有有限个，记为 $Q_1, \ldots, Q_t$.

(iii) $P$ 在每个 $Q_i$ 处的剩余类域次数 $f_i$ 是有限的．

(iv) $[L:K] = \sum_{i=1}^{t} e_i f_i$. 其中 $e_i$, $f_i$ 分别为 $P$ 在 $Q_i$ 处的分歧指数和剩余类域次数．

(v) 在 $B$ 中成立理想的等式 $PB = Q_1^{e_1} Q_2^{e_2} \ldots Q_t^{e_t}$.

**证明** (i): 由 $B$ 的定义知 $B$ 为整闭整环．由 $B$ 为有限 $A$-模知 $B$ 为 Noether 环．由定理 4.2.1 只需再证明 $B$ 为一维环．设 $P_1 \subset P_2$ 为 $B$ 的两个素理想．令 $P_0 = (0)$. 则

$$(0) = P_0 \cap A \subset P_1 \cap A \subset P_2 \cap A$$

为 $A$ 的三个素理想．而 $A$ 为一维环，故 $P_0 \cap A, P_1 \cap A, P_2 \cap A$ 中至少有两个相等．再由整扩张的纤维中的素理想没有包含关系 (命题 3.6.5) 知 $P_0, P_1, P_2$ 中至少有两个相等．由此得到 $B$ 为一维环．

(ii): 设素理想 $Q$ 满足 $Q \cap A = P$. 由 $P \neq (0)$ 知 $Q \neq (0)$. 通过在 $P$ 处作局部化，$A_P \hookrightarrow B_P$ 为单的整扩张，故 $B_P/PB_P$ 为零维 Noether 环．从而其素理想均为极小素理想，故只有有限个，这些素理想一一对应到 $B$ 中满足 $Q \cap A = P$ 的素理想 $Q$.

(iii) 和 (iv): 通过将 $A, B$ 分别替换为 $A_P, B_P$, 我们不妨设 $A$ 为 DVR, $P = m = (\pi)$ 为 $A$ 的唯一极大理想．则 $A \hookrightarrow B$ 为有限扩张．?由 $Q_i \cap A = m$ 知 $A/m \hookrightarrow B/Q_i$ 为单的有限扩张．从而 $[k(Q_i) : k(m)] = f_i$ 为有限数．

f.g + torsion free ⇒ free.

由于 $B$ 为有限 $A$-模，而 $A$ 为 DVR, 根据主理想整环上有限模的结构定理，可知 $B$ 为秩有限的自由 $A$-模，记 $n = \mathrm{rank}_A(B)$. 由 $A \hookrightarrow B$ 为单的有限扩张知 $K = A \otimes_A K \hookrightarrow B \otimes_A K$ 也为单的有限扩张，从而 $B \otimes_A K$ 为域．而作为 $B$ 的局部化，我们有自然的单同态 $B \subset B \otimes_A K \subset L = \mathrm{Frac}(B)$, 从而得到 $B \otimes_A K = L$, 故 $[L:K] = n$.

K flat

Hilbert Nullstellensatz

另一方面，记 $k = A/m$, 则由 $B$ 为秩 $n$ 的自由 $A$-模知 $B/mB = B \otimes_A k$ 为 $n$ 维 $k$-线性空间，即 $\dim_k B/mB = n$. 注意到 $B/mB$ 为零维 Noether 环，并且 $\mathrm{Spec}(B/mB) = \{Q_1, \ldots, Q_t\}$, 由命题 1.4.7 得到环同构

-scheme.

$$B/mB \simeq B_{Q_1}/mB_{Q_1} \times \cdots \times B_{Q_t}/mB_{Q_t}. \tag{4.3–1}$$

在离散赋值环 $B_{Q_i}$ 中，设 $Q_i B_{Q_i} = (\pi_i)$, 则 $mB_{Q_i} = (\pi_i)^{e_i}$. 从而 $B_{Q_i}/mB_{Q_i} = B_{Q_i}/(\pi_i)^{e_i}$. 由此得到

$$\begin{aligned} \dim_k\left(\frac{B_{Q_i}}{mB_{Q_i}}\right) &= \ell_A\left(\frac{B_{Q_i}}{mB_{Q_i}}\right) = \sum_{j=1}^{e_i} \ell_A\left(\frac{(\pi_i)^{j-1}}{(\pi_i)^j}\right) = \sum_{j=1}^{e_i} \ell_A\left(\frac{B_{Q_i}}{(\pi_i)}\right) \\ &= \sum_{j=1}^{e_i} \dim_k k(Q_i) = e_i f_i. \end{aligned}$$

化取



分别计算(4.3–1) 两边作为 $k$-线性空间的维数即得 $n = \sum_{i=1}^{t} e_i f_i$.

(v): 设 $Q$ 为 $B$ 的非零素理想, 我们比较 $P$ 和 $Q_1^{e_1}Q_2^{e_2}\ldots Q_t^{e_t}$ 在 $B_Q$ 中生成的理想, 只需证明 $PB_Q = Q_1^{e_1}Q_2^{e_2}\ldots Q_t^{e_t}B_Q$.

如果 $Q \cap A \neq P$, 则 $PB_Q = B_Q$, 而且 $Q \neq Q_i$, $\forall\ 1 \leq i \leq t$. 故对每个 $i$ 有 $Q_iB_Q = B_Q$. 这样得到 $PB_Q = Q_1^{e_1}Q_2^{e_2}\ldots Q_t^{e_t}B_Q = B_Q$.

如果 $Q \cap A = P$, 则存在某个 $i$ 使得 $Q = Q_i$. 由 $e_i$ 的定义得到 $PB_Q = Q_i^{e_i}B_Q$. 对 $j \neq i$, 由于 $Q_i$ 和 $Q_j$ 为不同的极大理想, 不难看到 $Q_j^{e_j}B_Q = B_Q$. 由此也得到 $PB_Q = Q_1^{e_1}Q_2^{e_2}\ldots Q_t^{e_t}B_Q$. □

---

Dedekind domain.

$\Leftrightarrow$ 1 dimensional, Noetherian, integral closed, integral domain.

---

Theorem.

$\mathbb{Z}[\zeta_N]$ is a dedekind domain.

① $N = p$ prime.

$\mathbb{Z}[\zeta_N]$ $\quad \mathfrak{p} \neq (0)$

$\mathbb{Z}$

$\Rightarrow \mathfrak{p} \cap \mathbb{Z} \neq (0)$, or $\mathfrak{p} \cap \mathbb{Z} = 0$,

$(0) \subsetneq \mathfrak{p}$, $\times$

$(\mathfrak{p} \cap \mathbb{Z}) = (q)$

(i) $q \neq p$.

$$(p)/(q) \subseteq \frac{\mathbb{Z}[\zeta_p]}{(q)}$$

$$\frac{\mathbb{Z}[x]/(x^p-1)}{(q)} \twoheadrightarrow \mathbb{Z}[\zeta_p]/(q)$$

$$\parallel$$

$$\mathbb{F}_q[x]/(x^p-1).$$

$x^p-1$ has no multiple roots.

$$\frac{\mathbb{F}_q[x]}{(x^p-1)} = \mathbb{F}_q[x]/(f_1 f_2 \cdots f_m)$$

$$= k_1 \times \cdots \times k_m$$

$$\Rightarrow \mathbb{Z}[\zeta_p]/(q) = k_1 \times \cdots \times k_r$$

$$\Rightarrow \left(\frac{\mathbb{Z}[\zeta_p]}{(q)}\right)_{\frac{\mathfrak{p}}{(q)}} \text{ is a field.}$$

$$\Rightarrow \frac{\mathfrak{p}}{(q)} \cdot \left(\frac{\mathbb{Z}[\zeta_p]}{(q)}\right)_{\frac{\mathfrak{p}}{(q)}} = (0)$$

$$\left(\frac{\mathbb{Z}[\zeta_p]}{(q)}\right) = \frac{\mathbb{Z}[\zeta_p]_{\mathfrak{p}}}{(q)}$$

$$\Rightarrow \mathfrak{p} = (q) \text{ in } \mathbb{Z}[\zeta_p]_{\mathfrak{p}}$$

(ii) $\mathfrak{p} \cap \mathbb{Z} = q\mathbb{Z}$

$$\frac{P}{(p)} \subseteq \frac{\mathbb{Z}[\zeta_p]}{(p)}$$

$$\frac{\mathbb{Z}[x]/(x^p - 1)}{(p)} = \frac{\mathbb{F}_p[x]}{(x^p - 1)}$$

$$= \frac{\mathbb{F}_p[x]}{(x-1)^p} \supseteq (x-1)$$

In $\mathbb{Z}[\zeta_p]$ $P = (p, \zeta_p - 1)$

$$\frac{x^p - 1}{x - 1} = (x-1)^{p-1} + p(x-1)^{p-2} + \cdots + p$$

$x = \zeta_p.$

$\Rightarrow \zeta_p - 1 \mid p$ in $\mathbb{Z}[\zeta_p]$

$$(\zeta_p-1)^{p-1} + p(\zeta_p-1)^{p-2} + \cdots + p = 0$$

$1+(\zeta_p-1)\square \notin (\zeta_p-1).$

In $\mathbb{Z}[\zeta_p]$

$(\zeta_p-1)$

$$p = u\cdot(\zeta_p-1)^{p-1}$$

$$\Rightarrow \quad p = (\zeta_p-1)^{p-1}$$

② $N = p^n$. similar.

$\mathbb{Z}[\zeta_{p^n}]$

$\mathbb{Z}$.

③. $N = p^m \cdot n$. $\quad (p,n)=1$, $m \geq 1$

$$\begin{tikzcd}
\mathbb{Z}[\zeta_N] = \mathbb{Z}[\zeta_n][\zeta_{p^m}] \ni \mathfrak{P} \neq (0) \arrow[d, dash] \\
\mathbb{Z}[\zeta_n] \arrow[d, dash] \\
\mathbb{Z}
\end{tikzcd}$$

$\mathbb{Z}[\zeta_n]$ Dedekind domain.

$$\mathfrak{P} \cap \mathbb{Z}[\zeta_n] = \mathfrak{P}_1$$

$$\mathfrak{P}_1 \cap \mathbb{Z} = \mathfrak{P}_2.$$

$A \to B \qquad p \qquad \{\mathfrak{q}_1, \sim \mathfrak{q}_m\}.$

$$pB_{\mathfrak{q}_i} = \mathfrak{q}_i^{e_i}$$

---

$$\begin{tikzcd} B \arrow[r, phantom, "\subseteq"] \arrow[d, dash] & L \arrow[d, dash] \\ A \arrow[r, phantom, "\subseteq"] & K \end{tikzcd}$$

① $L/K$ finite extension

② $\mathrm{Frac}(A) = K$

$\mathrm{Frac}(B) = L$

③ $A, B$ Dedekind.

④ $B$ f.g $A$ module.

Theorem.

$$\begin{array}{ccc} B & \subseteq & L \\ | & & | \\ A & \subseteq & K \end{array}$$

$L = \mathrm{Frac}(B)$ $K = \mathrm{Frac}(A)$

$A, B$ integral closed.

$L/K$ finite extension.

If (i) $L/K$ separable

(ii) $A$ is a f.g. $k$-algebra.

($k$ is any field)

Pf: (i) dual basis.

(ii)

$$\begin{array}{ccc} B & \subseteq & L \\ | & & | \\ A & \subseteq & K \\ | & & | \\ k[t_1,\dots,t_n] & \subseteq & k(t_1,\dots,t_n) \\ | & & \\ k & & \end{array}$$

finite.

Noetherian normalization.

We can suppose $A = k[t_1,\dots,t_n]$

char $k = p > 0$

$\boxed{\text{purely inseparable}}$.

$$\begin{array}{ccc} B & \subseteq & L \\ | & & | \\ B' & \subseteq & L_{sep} \end{array} \qquad \times .$$

Field theory: $L/K$ normal extension

$\Rightarrow \exists\ K - L_{insep} - L$

$L_{insep}/K$ purely inseparable

$L/L_{insep}^{(L_i)}$ separable.

$L_{insep} := L^{Aut(L/K)}$

$$
\begin{tikzcd}
B_i \arrow[r, phantom, "\subseteq"] \arrow[d, dash] & L_i \arrow[d, dash] \\
k[t_1, \cdots, t_k] \arrow[r, phantom, "\subseteq"] & K
\end{tikzcd}
$$

设 $L = K(b_1, \ldots, b_m)$, $b_i \in B, \forall\, 1 \le i \le m$. 由 $L/K$ 为纯不可分的有限扩张, 存在 $p$ 的一个正整数幂次 $q$, 使得 $b_i^q \in K, \forall\, 1 \le i \le m$. 又由于 $b_i^q$ 在 $A$ 上整, 并且 $A$ 为整闭的, 故每个 $b_i^q$ 均在 $A = k[t_1, \ldots, t_n]$ 中. 这样可以找到有限个 $k$ 中的元素 $c_1, \ldots, c_r$, 使得每个 $b_i$ 均在 $k'(t_1^{\frac{1}{q}}, \ldots, t_n^{\frac{1}{q}})$ 中, 其中 $k' = k(c_1^{\frac{1}{q}}, \ldots, c_r^{\frac{1}{q}})$ 为 $k$ 的有限扩域. 由此得到 $L \subset k'(t_1^{\frac{1}{q}}, \ldots, t_n^{\frac{1}{q}})$, 从而 $B$ 包含在 $A$ 在 $k'(t_1^{\frac{1}{q}}, \ldots, t_n^{\frac{1}{q}})$ 的整闭包 $B'$ 中. 不难看到 $B' = k'[t_1^{\frac{1}{q}}, \ldots, t_n^{\frac{1}{q}}]$. 这是因为 $k'[t_1^{\frac{1}{q}}, \ldots, t_n^{\frac{1}{q}}]$ 在 $A$ 上整, 同时 $k'[t_1^{\frac{1}{q}}, \ldots, t_n^{\frac{1}{q}}]$ 同构于 $k'$ 上的 $n$ 元多项式环, 从而为整闭整环. 由于 $A \hookrightarrow B'$ 显然为有限有限扩张, 从而由 Noether 性知 $B$ 为有限 $A$-模. $\square$

$\square$

Theorem.

$A$ is a Dedekind domain.

$\forall\, 0 \neq I \subseteq A$

$\exists!\ I = P_1^{a_1} \cdots P_m^{a_m}$

$0 \neq P_i \in \operatorname{Spec} A$.

Pf: $\forall\, 0 \neq P \in \operatorname{Spec} A$

$I A_P = P^{v_P(I)}$

---

Claim: $I = \prod_{P \in \operatorname{Spec} A} P^{v_P(I)}$

There are only finite many $p$ s.t. $I \subseteq p$ i.e. $v_p(I) \geq 1$

$$I_p = \Big(\prod_{p \in \operatorname{Spec} A} p^{v_p(I)}\Big)_p, \quad \forall p.$$

$$\{ I \mid \underset{\text{ideal}}{I \subseteq A} \quad I \neq 0 \} \supseteq \{ (a) \mid \begin{matrix} a \neq 0 \\ a \in A \end{matrix} \}.$$

ideal class group:

- line bundle

- divisor
- group of ideal class

Definition.

$A$ Noetherian ring.

$M$ is an $A$-module, if

$\forall \mathfrak{p} \in \operatorname{Spec} A$

$M_{\mathfrak{p}} \xrightarrow{\sim} A_{\mathfrak{p}}$ (locally free)

call $M$ a invertible module

(or locally free module of rank 1)

line bundle.

e.g. $A$ Dedekind domain

$0 \neq I \subseteq A$ ideal

$\Rightarrow I$ is an invertible module.

$$I_p = IA_p = (\pi^n) \xrightarrow{\sim} A_p.$$

$$\{\text{Invertible } A\text{-module}\} / \sim$$

isomorphism.

$$\| $$

$$[M] \in \mathrm{Pic}(A)$$

$[M]\cdot[N] = [M\otimes N]$

$(M\otimes_A N)_p = M_p \otimes_{A_p} N_p$

Prop. $\forall M \in \mathrm{Pic}(A)$

$\exists N$, s.t.

$M\otimes_A N \xrightarrow{\sim} A$

$\mathrm{Hom}(W, A\otimes V)$

$\downarrow s$

$\mathrm{Hom}(W, V)$

$(\phi\otimes y)(w)$

$\parallel$

$\phi(w)$

$\boxed{N = \mathrm{Hom}(M, A)}$

$M\otimes N \xrightarrow{\sim} A$

$\underbrace{\phantom{M\otimes N \xrightarrow{\sim} A}}$

$m\otimes\phi \longmapsto \phi(m)$

$$\mathrm{Hom}_A(M, A) \otimes A_p = \mathrm{Hom}_{A_p}(M_p, A_p)$$

$$\xrightarrow{\sim} A_p.$$

Proposition.

$A$ is a dedekind domain.

$K = \mathrm{Frac}(A)$.

fractional ideal ( f.g. submodule of $K$)

is invertible

$\square$

$\Rightarrow$ {fractional ideals} $\subseteq$ {invertible ideals}.

Proposition.

A is dedekind.

M is an invertible A-module

$\Rightarrow \exists$ A module injection

$M \hookrightarrow K$

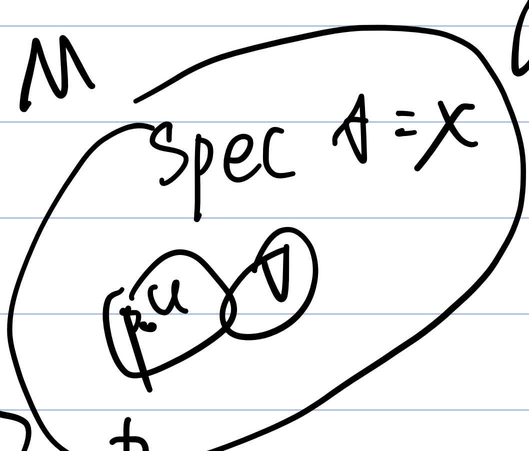
pf: $M \to M_p \xrightarrow{\sim} A_p \hookrightarrow K$

$p \in U \subseteq \operatorname{Spec} A$.

injective?

It's enough to prove

$\tilde{M}(V)$




Prop. $I_1, I_2$ fractional ideal

$I_1 \xrightarrow{\sim} I_2 \iff \exists x \in K^*,$

$I_1 = x I_2$

Proposition. $\begin{matrix} \text{f.g} \\ \Updownarrow \end{matrix}$

$M$ invertible

$\Rightarrow \forall p$, $\exists f$, s.t. $p \in D(f)$

$M_f \xrightarrow{\sim} A_f$ $\left( \begin{matrix} M_p \xrightarrow{\sim} A_p \\ \text{induces } A \xrightarrow{\varphi} M \\ \text{induces } A_f \xrightarrow{\varphi_f} M_f \end{matrix} \right)$

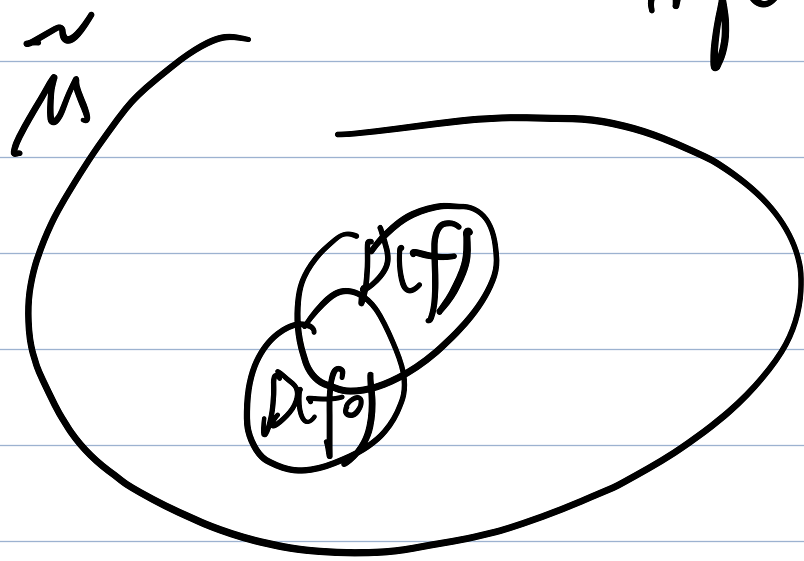
Proposition. $M$ invertible

$A$ Noetherian, integral

$\Rightarrow M \hookrightarrow \mathrm{Frac}(A)$

$M \longrightarrow M_f \hookrightarrow K$

injective :




$$\{\text{invertible ideals}\}/\sim \; = \mathrm{Pic}(A)$$

$$\big\downarrow \wr$$

$$\{\text{fractional ideals}\}/K^*$$

$$\{\text{fractional ideal}\} \quad I \cdot J \cong I \otimes J$$

$$I^{-1} = \mathrm{Hom}(I, A) \hookrightarrow K$$

$$\xrightarrow{\sim} \{x \mid ax \in A, \forall a \in I\}$$

$$IA_{p_i} = P_i^{\alpha_i} \quad \text{in} \quad K.$$

$$\begin{array}{ccc} M_f & \longrightarrow & M_{fg} \\ \downarrow \wr & & \downarrow \wr \\ A_f & \hookrightarrow & A_{fg} \end{array}$$

$$IA_p = \left(\pi_p^{\,V_p(I)}\right) \cdot A_p$$

$$\Rightarrow I = \prod_{p \in \operatorname{Spec} P} p^{v_p(I)}$$

$$\operatorname{Div}(A) := \bigoplus_{p \in \operatorname{Spec} A} \mathbb{Z}^{p} \qquad \operatorname{div}(I) := \sum_{p} p\, v_p(I)$$

$$\begin{array}{c} \operatorname{Cl}(A) \xrightarrow{\sim} \operatorname{Div}(A) \big/ \{\operatorname{div}(f) / f \in K\} \\ \downarrow\wr \\ \operatorname{Pic}(A) \end{array}$$

Gauss's conjecture.

real quadratic field

$K = \mathbb{Q}(\sqrt{n})$ $(n \geq 1,$ $n$ square free)

$\Rightarrow Cl(\mathcal{O}_K)$ is trivial.

---

Dimension.

Definition.

Krull dimension of $A$ is

$\sup \{ n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \}$

Theorem 1. (going-up theorem)

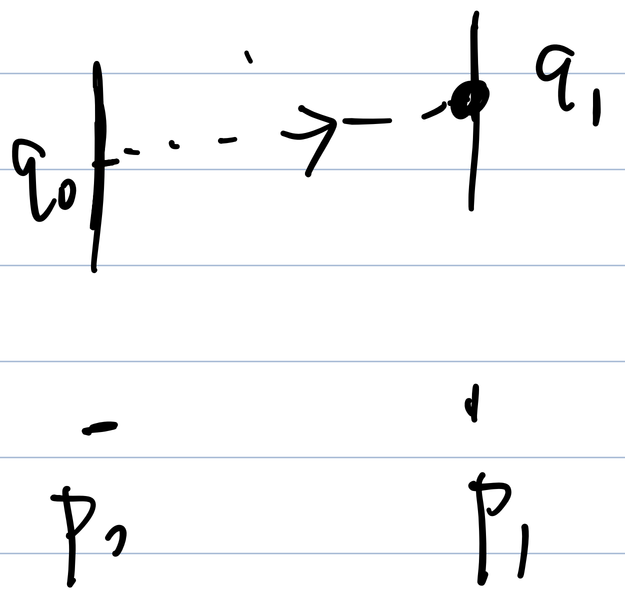
$A \to B$ integral extension, $P_0 \subsetneq P_1$

$P_i \in \operatorname{Spec} A$

$\Rightarrow \forall\, q_0 \in \operatorname{Spec} B$, s.t. $q_0 \cap A = P_0$

$\exists\, q_1 \in \operatorname{Spec} B$, s.t. $q_1 \cap A = P_1$

$q_0 \subsetneq P_1$




Theorem 2 (Going-down theorem)

$A \to B$ is injective extension

of integral domains, $A$ is integral

closed

$\Rightarrow$ The going down theorem

hold:

$\forall\, \mathfrak{p}_0, \mathfrak{p}_1 \in \operatorname{Spec} A$

$\forall\, \mathfrak{q}_1 \in \operatorname{Spec} B, \quad \mathfrak{q}_1 \cap A = \mathfrak{p}_1$

$\exists\, \mathfrak{q}_0 \in \operatorname{Spec} B, \quad \mathfrak{q}_0 \cap A = \mathfrak{p}_0$

$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1$

$\mathfrak{q}_0 \longleftarrow \mathfrak{q}_1$

$P_0$ $\quad$ $P_1$

Theorem (Noether Normalization Lemma).

Let $k$ be a field.

$A$ is a f.g. $k$-algebra.

$\forall\ I_0 \subseteq \cdots \subseteq I_m$ be an ideal chain of $A$

$\Rightarrow \exists\ t_1 \sim t_n \in A$, s.t.

(1) $t_1 \sim t_n$ are algebraic independent

(2) $k[t_1, \cdots, t_n] \hookrightarrow A$ is a finite extension

(3) $\forall i$,

$I_i \cap k[t_1, \cdots, t_n] = (t_1, \cdots, t_{k(i)})$

$I_1 \subseteq I_2 \subseteq \cdots\cdots$

$\Downarrow$

$(t_1) \subseteq (t_1, t_2) \subseteq \cdots\cdots$

Proposition.

$\dim k[t_1, \cdots, t_n] = n.$

Pf: use Noether normalization

theorem.

Proposition.

$k \hookrightarrow A$ $A$ f.g. integral $k$

algebra

$\Rightarrow \dim A = \text{tr.d. } \mathrm{Frac}(A)$

Pf:

$$\begin{array}{c} A \\ \uparrow \text{ finite} \\ k[t_1, \dots, t_n] \end{array}$$

going up + Noether normalization thm. $\square$

**定理 5.2.1** 设 $A$ 为域 $k$ 上的有限生成代数，且为整环. 令 $K = \mathrm{Frac}(A)$ 为 $A$ 的分式域. 则有:

(i) $\dim A = \mathrm{trdeg}_k K$. 其中 $\mathrm{trdeg}_k K$ 为 $K/k$ 的超越次数 ([14, Definition 030G]).

(ii) $A$ 的任意素理想链 $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r$ 均可扩充为一个饱和素理想链.

(iii) $A$ 的任意两个饱和素理想链的长度相等.

(iv) 对 $A$ 的任意极大理想 $m$, 有 $\dim A = \dim A_m$.

**证明** 任取 $A$ 的素理想链 $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r$. 由理想链形式的 Noether 正规化定理 5.1.3, 可以找到在 $k$ 上代数无关的元 $t_1, \ldots, t_n$, 以及非负整数 $0 \le h(0) \le h(1) \le \cdots \le h(r) \le n$, 使得 $k[t_1, \ldots, t_n] \hookrightarrow A$ 为整扩张, 并且 $P_i \cap k[t_1, \ldots, t_n] = (t_1, \ldots, t_{h(i)})$. 由于整扩张每个纤维中的素理想没有包含关系, 我们看到 $h(i) < h(i+1)$, $\forall\ i = 0, \ldots, r-1$. 由下降定理 5.1.2, $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r$ 可以扩充为素理想链 $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$, 使得 $Q_i \cap k[t_1, \ldots, t_n] = (t_1, \ldots, t_i)$, $\forall\ i = 0, \ldots, n$. 注意到 $k[t_1, \ldots, t_n]$ 中的素理想链 $(0) \subsetneq (t_1) \subsetneq (t_1, t_2) \subsetneq \cdots \subsetneq (t_1, \ldots, t_n)$ 已经饱和, 再由整扩张每个纤维中的素理想没有包含关系, 我们看到素理想链 $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$ 是饱和的. 这样我们证明了 $A$ 的任何有限长素理想链都可以扩充为长度为 $n$ 的饱和素理想链. 由 $n = \mathrm{trdeg}_k K$, 定理得证. □

Noetherian local ring.

$(A, m, k)$ $\quad k = A/m$.

Definition.

$A$ 的一个参数系是一组元素 $x_1 \sim x_n \in m$,

s.t. $\exists\, i \geq 1$, $m^i \subseteq (x_1, \cdots, x_n)$

e.g. ① $A = k[x]/(x^2)$

$m = (\bar{x})$

$\phi$ 为参数系.

极小参数系：集合包含意义下

Theorem. $(A, m)$ Noetherian local ring

$\Rightarrow$ 任取极小参数系 $\{x_0, \cdots, x_n\}$

$\Rightarrow \dim A = n$

i.e. $|$极小参数列$| - 1 = \dim A$.

Corollary.

$(A, m)$ Noetherian local ring

$$\dim A \leq \underbrace{\dim_k m/m^2}_{\text{cotangent space}}.$$

If $\dim A = \dim_k m/m^2$

Call $A$ a regular local ring

Integrally closed: normal.

Proposition.

$k$ is a field

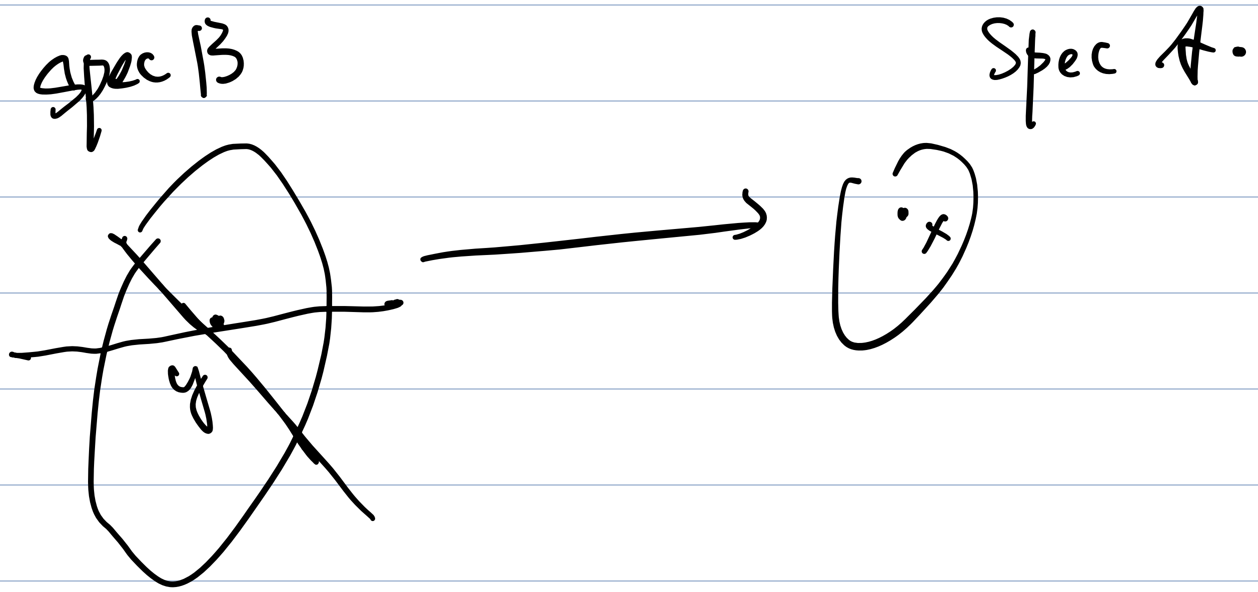
$f: A \to B$ is a homomorphism of f.g. $k$-algebra, and is injective, $A$ is integral.

$y \in \operatorname{Spec} B$ is closed.

$\Rightarrow$ (1) $x = f^{-1}(y) \in \operatorname{Spec} A$ is closed

(2) $Z$ is any irreducible component containing $y$ of $(f^*)^{-1}(x)$

$\Rightarrow \dim Z \geq \dim B - \dim A$.




Pf:

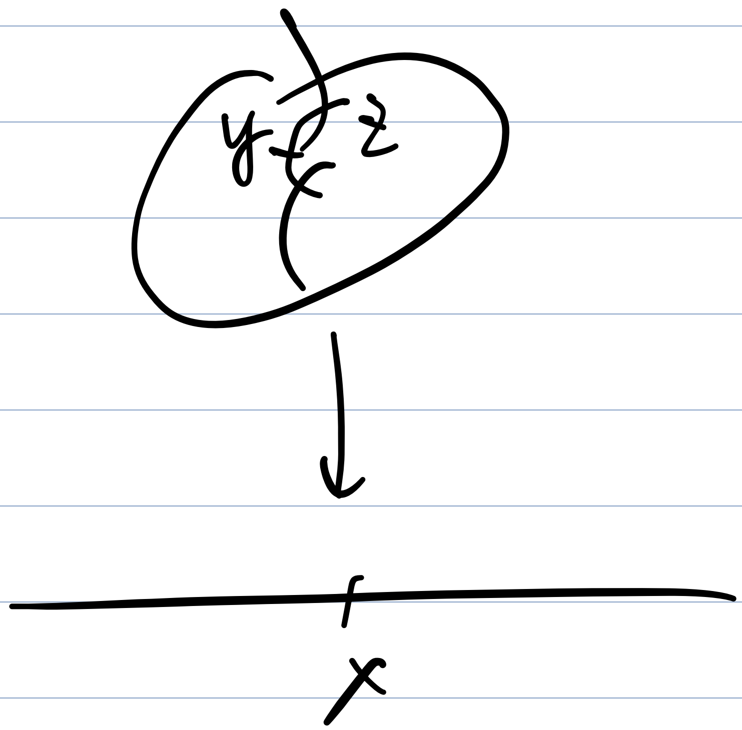
(1) $\begin{tikzcd} k \arrow[r, hook] \arrow[rr, bend left, dashed, "\text{finite algebraic}"] & K(x) \arrow[r, hook] & K(y) \end{tikzcd}$

x is a closed point

$\Leftrightarrow$ $k \hookrightarrow K(x)$ is a finite algebraic extension.

(2) 取A的饱和素理想链

$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$, $n = \dim A$.




$$f^{-1}(x) \cong \operatorname{Spec} B \otimes_A A/\mathfrak{m}_x$$

$$Z \xrightarrow{\sim} \operatorname{Spec} \frac{B \otimes_A A/\mathfrak{m}_x}{\mathfrak{p}}$$

$$= \operatorname{Spec} \frac{B/m_x B}{P} \qquad (m_x, P)$$

$$= \operatorname{Spec} \left( \frac{B/m_x B}{P} \right)_m$$

Reduce to local ring

(3) $\exists\, \phi \neq U \subseteq \operatorname{Spec} A$, s.t.

$U \subseteq \operatorname{Im} f$, $\forall x \in U$, $\forall$

irreducible component of $f^{-1}(x)$ $Z$

has
$$\dim Z = \dim B - \dim A$$

Thm.

$A \hookrightarrow B$ is an injection

between f.g. $k$-algebra, $A, B$

are integral domain. $Y = \operatorname{Spec} B \xrightarrow{f} X = \operatorname{Spec} A$

(1) $y \in Y$ is closed $\Rightarrow$ $x = f(y)$

is closed

(2) $x \in X$ is closed, if $f^{-1}(x) \neq \emptyset$

$\Rightarrow$ Every irreducible component $Z$,

$\dim Z \geq \dim Y - \dim X$

(3) $\exists$ non-empty open set $U \subseteq X$,

s.t. $\forall$ closed pt $x \in U$,

$f^{-1}(x) \neq \emptyset$, and

every irreducible component $Z$ of $f^{-1}(x)$, $\dim Z = \dim Y - \dim X$

Pf of (3):

Noetherian Normalisation:

$$A \hookrightarrow A[X_1, \cdots, X_n] \hookrightarrow B$$

Substitute $A$ to $A_q$.

$A \otimes_k k[X_1, \cdots, X_n]$

$$\begin{tikzcd} Y \arrow[d, "f"'] \arrow[r, "\text{finite type}"] & \operatorname{Spec} A[X_1, \cdots, X_n] = X \times \mathbb{A}^n_k \arrow[dl] \\ X & \end{tikzcd}$$

$P\, A[X_1, \cdots, X_n]$

$\Rightarrow \quad U = \operatorname{Spec} A_g$

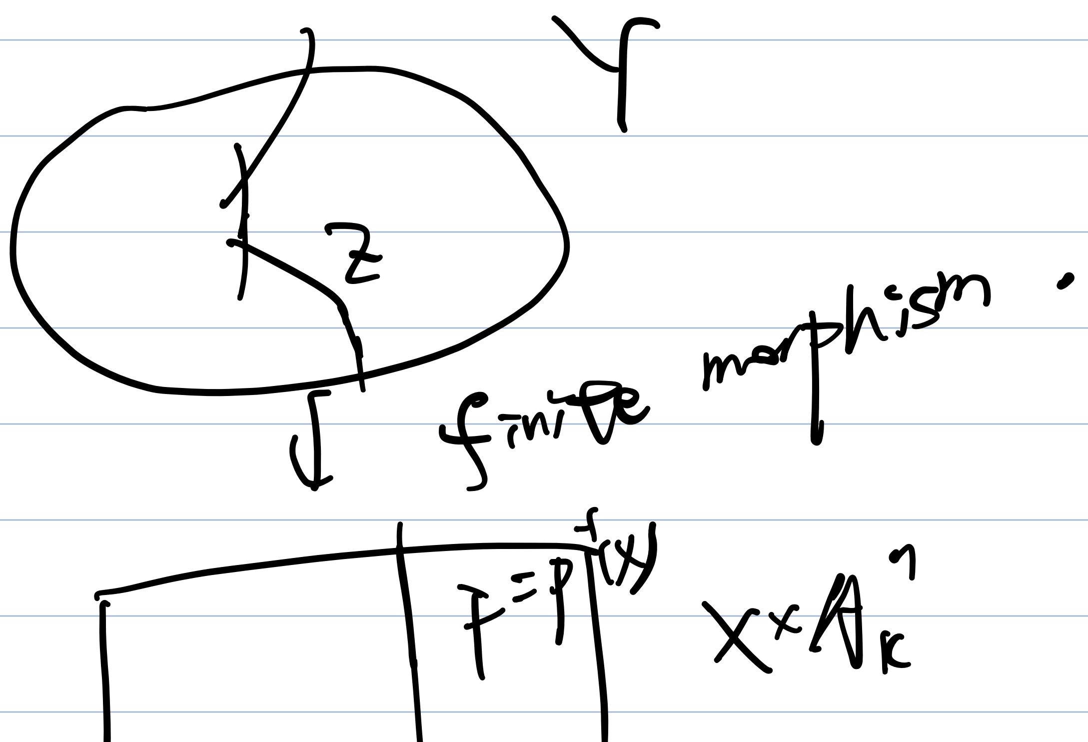
**定理 3.6.5 (Noether 正规化定理: 整环形式)** 设 $R$ 为 Noether 整环, $A$ 为有限生成 $R$-代数, 并且 $A \otimes_R \operatorname{Frac}(R) \neq 0$. 则存在 $0 \neq f \in R$, 同时满足:

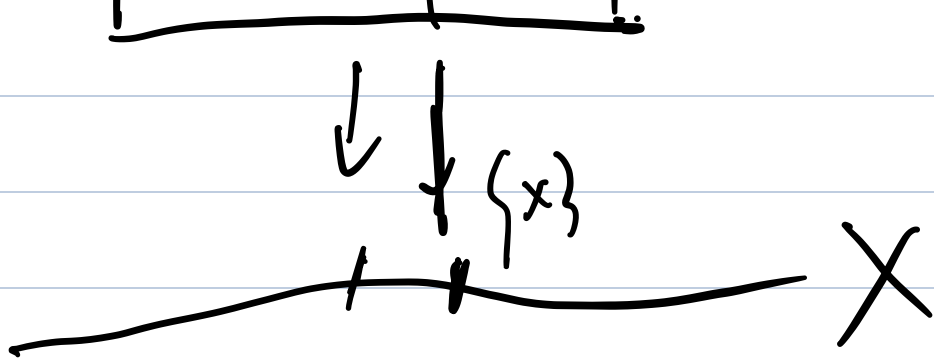
(i) $R_f \hookrightarrow A_f$ 为单同态.

(ii) 存在环同态的分解 $R_f \hookrightarrow B \hookrightarrow A_f$, 使得 $B$ 作为 $R_f$-代数同构于多项式代数 $R_f[x_1, \ldots, x_n]$, 以及 $B \hookrightarrow A_f$ 为有限扩张.

**证明** 设 $I$ 为结构同态 $R \to A$ 的核. 由 $A \otimes_R \operatorname{Frac}(R) \neq 0$ 知 $R \otimes_R \operatorname{Frac}(R) \to A \otimes_R \operatorname{Frac}(R)$ 为单同态, 从而 $I \otimes_R \operatorname{Frac}(R) = 0$. 由此知 $I = 0$. 故 $R \hookrightarrow A$ 为单同态.

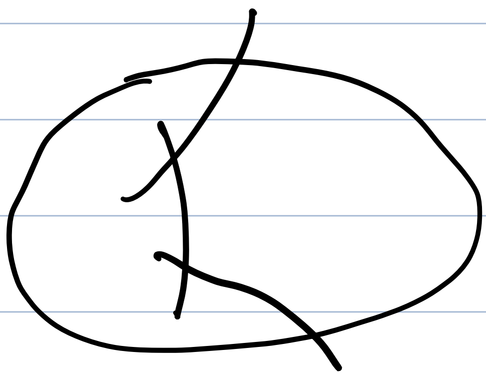
设 $A = R[a_1, \ldots, a_m]$. 对域 $\operatorname{Frac}(R)$ 上的有限生成代数 $A \otimes_R \operatorname{Frac}(R)$ 应用 Noether 正规化定理, 可以找到 $A \otimes_R \operatorname{Frac}(R)$ 中在 $\operatorname{Frac}(R)$ 上代数无关的元 $t_i$, $i = 1, \ldots, n$, 使得 $\operatorname{Frac}(R)[t_1, \ldots, t_n] \to A \otimes_R \operatorname{Frac}(R)$ 为整扩张. 由局部化的定义, 我们可以找到 $0 \neq f \in R$, 使得每个 $t_i$ 都在 $A_f$ 中, 并且每个 $a_i$ 满足系数在 $R_f[t_1, \ldots, t_n]$ 上的首一方程. 这样即知 $R_f[t_1, \ldots, t_n] \to A_f = R_f[a_1, \ldots, a_m]$ 为整扩张, 进而为有限扩张. 由 $t_1, \ldots, t_n$ 在 $\operatorname{Frac}(R)$ 上代数无关知 $R_f[t_1, \ldots, t_n]$ 作为 $R_f$-代数同构于 $R_f[x_1, \ldots, x_n]$. □



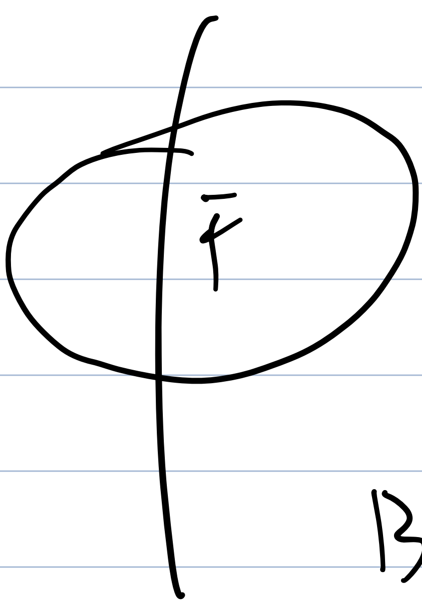



$\mathfrak{m} \subseteq A$.

$$\dim \bar{F} = \dim Y - \dim X$$



$\operatorname{Spec}(B/\mathfrak{p}B)$.




$\bar{F} = \operatorname{Spec} A[X_1, \dots, X_n]/P$

$B/(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n) =$

$$\mathrm{Spec}(B/\mathfrak{p}B) = \mathrm{Spec}(B/\mathfrak{q}_1) \cup \cdots\cdots$$
$$\cup\, \mathrm{Spec}(B/\mathfrak{q}_n).$$

$$\frac{A[X_1,\cdots,X_n]}{\mathfrak{p}} \xrightarrow{\text{finite}} B_{\mathfrak{q}}$$

We need to prove this is injective.

$$\Leftarrow\; \mathfrak{q} \cap A[X_1,\cdots,X_n] = \mathfrak{p}.$$

$$\mathfrak{p} \subseteq \mathfrak{q} \cap A[X_1,\cdots,X_n]$$

$\mathfrak{q}$ is a minimal prime ideal

containing $\mathfrak{p}$

If $q \cap A[x_1, \dots, x_n] = P_1$

$q$.

$P$ — $P_1$

If going-down then hold,

we get a contradiction!

$A[x_1, \dots, x_n]$ is integrally closed?

Lemma.

$A$ is an integrally closed domain

$\Rightarrow$ $A[x]$ is

an integrally closed domain

Pf: $A = \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_\mathfrak{p}$ $\quad A_\mathfrak{p}$ is DVR

$\Rightarrow A[x] = \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_\mathfrak{p}[x]$

$\subseteq \operatorname{Frac}(A)[x]$

$A_\mathfrak{p}$ DVR $\Rightarrow$ $A_\mathfrak{p}$ UFD

$\Rightarrow A_\mathfrak{p}[x]$ UFD

$\Rightarrow A_\mathfrak{p}[x]$ Integrally closed.

$\Rightarrow A[x] = \bigcap A_p[x]$ is integrally closed.

Lemma, $A$ is a f.g. $k$-algebra, and is integral.

$\Rightarrow \exists\, 0 \neq g \in A$, s.t.

$A_g$ is integrally closed.

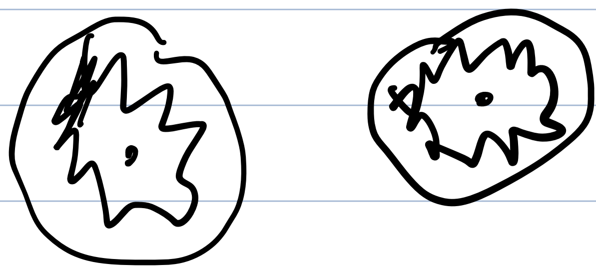
Pf:

$$\begin{tikzcd}
 & \mathrm{Frac}(A) & \\
A \arrow[ur, hook] \arrow[rr, hook] & & \bar{A} \arrow[ul, hook']
\end{tikzcd}$$

where $\bar{A}$ is the integral closure.

$$S = A \setminus \{0\}$$

$$S^{-1}A = S^{-1}\bar{A} = K$$



It's enough to prove $\bar{A}$ is a f.g.

$A$-module.

This is True!

\begin{tikzcd}
A \arrow[r, phantom, "\subseteq"] \arrow[d, dash] & K \arrow[d, dash] \\
A \arrow[r, phantom, "\subseteq"] & K
\end{tikzcd}

$\bar{A} = A[x_1, \dots, x_n]$

$g := x_1 \cdots x_n$

$\Rightarrow A_g = \bar{A}_g$ is integrally

Closed.

$\square$

Thm.

$A \hookrightarrow B$ is an injection

between f.g. $k$-algebra, $A, B$

are integral domain. $Y = \operatorname{Spec} B \xrightarrow{f} X = \operatorname{Spec} A$

(1) $\exists$ non-empty open set $u \subseteq X$,

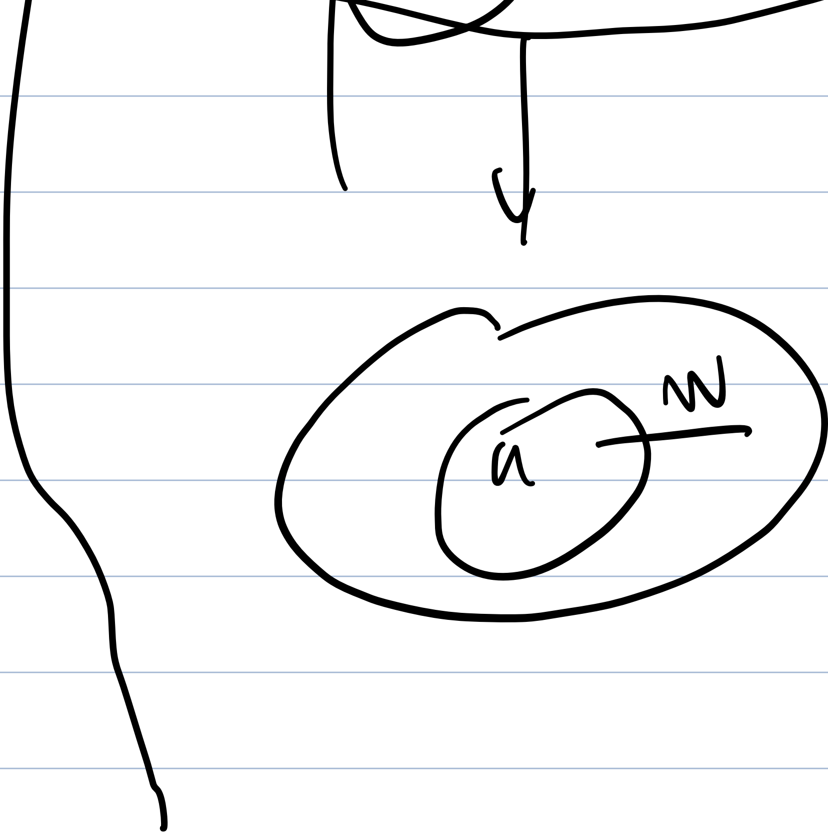
s.t. $u \subseteq f(Y)$, $\forall$ irreducible subset

$W \subseteq X$, $f^{-1}(W)$ is irreducible, if

If $W \cap u \neq \emptyset$, $Z \cap f^{-1}(u) \neq \emptyset$

$\Rightarrow \dim Z = \dim W + \dim Y - \dim X$.

$Z$ is an irre. component of $W$.




$$\begin{tikzcd}
\bar{A} \arrow[r, dash] \arrow[dr, dash, dashed] \arrow[dd, dash, bend right] & K = \mathrm{Frac}(A) \arrow[dd, dash, "\text{finite extension}"] \\
& A \arrow[dl, dash] & \\
k[t_1, \dots, t_n] \subseteq & k(t_1, \dots, t_n)
\end{tikzcd}$$

Hypersurface

$f \in k[x_1, \cdots, x_n]$ $\quad f \neq 0$

$V(f) \subseteq \mathbb{A}^n_k$

$\Rightarrow$ Every irre. component of $V(f)$

has $\dim = n-1$

$f = p_1 \cdots p_n$

$(f) = (p_1) \cap \cdots \cap (p_n)$.

height 1

每极小素理想

局部在极大理想处不影响

维数（对代数簇）

相当于取任何一分支.

Proposition

$(A, \mathfrak{q})$ is a Noetherian local ring

$f_1 \sim f_m \in \mathfrak{q}$.

$$\Rightarrow \dim \frac{A}{(f_1, \cdots, f_m)} \geq \dim A - m.$$

Pf:

设 $\bar{a}_1, \cdots, \bar{a}_d$ 为 $\dfrac{A}{(f_1, \cdots, f_m)}$ 的一组参数系.

$\Rightarrow a_1, \cdots, a_d, f_1, \cdots, f_m$ 为 $A$ 参数系

$\Rightarrow \dim A \leq d+m.$

$\square$

---

Regular local ring (正则局部环)

Definition. $A$ Noetherian, local.

$(A, m, k)$ is a <u>regular</u> local ring

$\Leftrightarrow \dim A = \dim_k m/m^2$

Example.

$k[X_1, \cdots, X_n]$ localize at $m \subseteq k$ is regular.

pf: $k = \bar{k}$:

$m = (X_1 - a_1, \cdots, X_n - a_n)$

$m/m^2 = k\,\overline{(X_1 - a_1)} + \cdots + k\,\overline{(X_n - a_n)}$

localize at $m$

$k \neq \bar{k}$

Pf:

$$k[x_1, \cdots, x_n] \hookrightarrow \bar{k}[x_1, \cdots, x_n]$$

$$m \qquad\qquad m'$$

$$m/m^2 \qquad\qquad m'/m'^2$$

$$m/m^2 \otimes_k \bar{k} \xrightarrow{\sim} m'/m'^2$$

Proposition.

$(A, m, k)$ regular.

$\bar f_1, \cdots, \bar f_k \in \frac{m}{m^2}$ are linear independent.

$\Rightarrow \frac{A}{(f_1, \cdots, f_k)}$ is regular.

Pf: $n = \dim A = \dim_k \frac{m}{m^2}$

$\dim \frac{A}{(f_1, \cdots, f_d)} \geq n-d$

$$\frac{\bar m}{\bar m^2} = \frac{\frac{m}{(f_1, \cdots, f_d)}}{\frac{m^2 + (f_1, \cdots, f_d)}{(f_1, \cdots, f_d)}} = \frac{m}{m^2 + (f_1, \cdots, f_d)}$$

$$= \frac{m/m^2}{(f_1, \cdots, f_d) + m^2}$$

$$\frac{(f_1, \ldots, f_d)}{m^2}$$

$$= \frac{m/m^2}{(\bar{f}_1, \ldots, \bar{f}_d)}$$

$$\Rightarrow \dim_k \frac{\bar{m}}{\bar{m}^2} = n - d.$$

$$\square$$

$$\dim_k \frac{\bar{m}}{\bar{m}^2} \geq \dim \bar{A}.$$

Proposition.

$(A, m, k)$ is regular.

$I \subseteq m \subseteq A$ s.t. $\bar{A} = A/I$

is regular

$\Leftrightarrow$ $I$ can be generated by

$I=(f_1,\cdots,f_d)$, s.t. $\bar{f}_1,\cdots,\tilde{f}_d \in \frac{m}{m^2}$

are linear independent.

pf: $n=\dim A$

$n-d=\dim A/I$

$\bar{m} = m/I$

$$\frac{\bar{m}}{\bar{m}^2} = \frac{m/I}{(m^2+I)/I} = \frac{m}{m^2+I}$$

$$= \frac{m/m^2}{(m^2+I)}$$

$$\frac{(m^2+I)}{m^2}$$

$\Rightarrow \exists\, f_1, \cdots, f_d \in I$, s.t.

$\overline{f_1}, \cdots, \overline{f_d}$ form a basis of

$k$- vector space $\dfrac{(m^2+I)}{m}$

$$I + m^2 \subseteq (f_1, \cdots, f_d) + m^2$$

we get

$$\frac{A}{(f_1, \cdots, f_d)} \twoheadrightarrow \frac{A}{I}$$

This is a surj. between

regular local rings of same dimension.

Lemma. Regular local ring are

**命题 5.4.2** 正则局部环均为整环.

**证明** 设 $(A, m)$ 为正则局部环. 对 $\dim A$ 归纳. 当 $\dim A = 0$ 时 $A$ 为域, 也为整环. 设 $\dim A > 0$. 取 $x \in m - m^2$, 并且 $x$ 不在任何极小素理想中. 则 $A/xA$ 为正则局部环. 由归纳假设, $A/xA$ 为整环. 从而 $(x)$ 为 $A$ 的素理想. 设 $P$ 为包含在 $(x)$ 中的极小素理想. 对任意 $a \in P$, 由 $P \subset (x)$ 知存在 $b \in A$ 使得 $a = bx$. 再由 $x \notin P$ 知 $b \in P$. 这样得到 $P = xP$. 由 Nakayama 引理知 $P = 0$. 故 $A$ 为整环. □

$P \subseteq (x)$  $a = bx$

素避引理.

素避引理 (prime avoidance).

**Statement**: Let $E$ be a subset of $R$ that is an additive subgroup of $R$ and is multiplicatively closed. Let $I_1, I_2, \ldots, I_n, n \geq 1$ be ideals such that $I_i$ are prime ideals for $i \geq 3$. If $E$ is not contained in any of $I_i$'s, then $E$ is not contained in the union $\cup I_i$.

**Proof by induction on $n$**: The idea is to find an element that is in $E$ and not in any of $I_i$'s. The basic case $n = 1$ is trivial. Next suppose $n \geq 2$. For each $i$, choose

$$z_i \in E - \cup_{j \neq i} I_j$$

where the set on the right is nonempty by inductive hypothesis. We can assume $z_i \in I_i$ for all $i$; otherwise, some $z_i$ avoids all the $I_i$'s and we are done. Put

$$z = z_1 \ldots z_{n-1} + z_n .$$

Then $z$ is in $E$ but not in any of $I_i$'s. Indeed, if $z$ is in $I_i$ for some $i \leq n-1$, then $z_n$ is in $I_i$, a contradiction. Suppose $z$ is in $I_n$. Then $z_1 \ldots z_{n-1}$ is in $I_n$. If $n$ is 2, we are done. If $n > 2$, then, since $I_n$ is a prime ideal, some $z_i, i < n$ is in $I_n$, a contradiction.

Cor. $(A, m)$, $\dim A > 0$

$\Rightarrow \exists x \in m \setminus m^2$,

$x$ is not contained in any minimal prime

Going down theorem.

$\boxed{?} \subseteq \mathfrak{q}_1$ $\qquad\qquad$ $\mathfrak{P}$

单的整扩张

A, B 整环

$P_0 \subseteq P_1$ $\qquad$ $A$

A 整闭.

Reduce to: $K(B)/K(A)$

finite, Galois, $B$ is the

integral closure.

$$G = \mathrm{Gal}(K(B)/K(A))$$

理想链 $P_0 \subset P_1 \subset \cdots \subset P_n$. 对每个 $i = 0, \dots, n-1$, 由 $Q_i \subsetneq Q_{i+1}$ 和命题 3.6.5 知 …… 结合已得

□

**定理 5.1.2 (下降定理)** 设 $A \hookrightarrow B$ 为整环的单同态, 并且为整扩张. 还设 $A$ 为整闭整环. 设 $P_1 \subsetneq P_2$ 为 $A$ 的素理想, $Q_2 \in \operatorname{Spec} B$ 并且 $Q_2 \cap A = P_2$. 则存在 $Q_1 \in \operatorname{Spec} B$ 满足 $Q_1 \cap A = P_1$, 以及 $Q_1 \subsetneq Q_2$.

**证明** 令 $K = \operatorname{Frac}(A)$, $L = \operatorname{Frac}(B)$, 则 $L/K$ 为代数扩张. 下面将问题逐步进行约化.

(1) 可设 $B$ 为 $A$ 在 $L$ 中的整闭包. 理由为: 令 $\tilde{B}$ 为 $A$ 在 $L$ 中的整闭包, 则有整扩张 $A \subset B \subset \tilde{B}$. 由上升定理, 可取 $\tilde{Q}_2 \in \operatorname{Spec} \tilde{B}$, 使得 $\tilde{Q}_2 \cap B = Q_2$. 如果对 $A \subset \tilde{B}$ 的情形已经证明了下降定理, 则可取 $\tilde{Q}_1 \in \operatorname{Spec} \tilde{B}$, 使得 $\tilde{Q}_1 \subset \tilde{Q}_2$, 且 $\tilde{Q}_1 \cap A = P_1$. 这样取 $Q_1 = \tilde{Q}_1 \cap B$ 即可.

(2) 可设 $L/K$ 为有限扩张. 理由为: 假设对有限扩张的情形已经证明了下降定理, 对 $L/K$ 的每个中间域 $M$, 记 $A_M$ 为 $A$ 在 $M$ 中的整闭包. 考虑如下集合

$$S \coloneqq \{(M, Q_M) \mid M \text{ 为 } L/K \text{ 的中间域}, Q_M \in \operatorname{Spec} A_M, Q_M \cap A = P_1, \\ Q_M \subset Q_2 \cap A_M \}.$$

定义 $S$ 上的偏序关系 $\le$ 为: $(M_1, Q_{M_1}) \le (M_2, Q_{M_2}) \iff M_1 \subset M_2$, 且 $Q_{M_2} \cap A_{M_1} = Q_{M_1}$. 对 $S$ 中的任意链 (全序子集) $\{(M_i, Q_{M_i}) \mid i \in I\}$, 令 $M \coloneqq \bigcup_{i \in I} M_i$, $Q_M \coloneqq \bigcup_{i \in I} Q_{M_i}$, 易验证 $(M, Q_M) \in S$, 并且 $(M_i, Q_{M_i}) \le (M, Q_M)$, $\forall\, i \in I$. 这说明 $S$ 中的任意链均有上界. 由 Zorn 引理, 可以找到 $S$ 的一个极大元 $(M_0, Q_{M_0})$. 如果 $M_0 \neq L$, 取 $a \in L \setminus M_0$, 令 $M_1 = M_0(a)$. 则 $M_1/M_0$ 为域的有限扩张, 并且对这个有限扩张应用下降定理可找到 $Q_{M_1} \in \operatorname{Spec} A_{M_1}$, 使得 $Q_{M_1} \cap A_{M_0} = Q_{M_0}$, 以及 $Q_{M_1} \subset Q_2 \cap A_{M_1}$. 这样得到 $(M_0, Q_{M_0}) < (M_1, Q_{M_1})$, 与 $(M_0, Q_{M_0})$ 的极大性矛盾. 故 $M_0 = L$, 从而取 $Q_1 = Q_{M_0}$ 即可.

(3) 可设 $L/K$ 为正规扩张. 理由为: 取 $\tilde{L}$ 为 $L/K$ 的正规闭包, 令 $\tilde{B}$ 为 $A$ 在 $\tilde{L}$ 中的整闭包. 则 $A \subset B \subset \tilde{B}$ 为整扩张. 由上升定理, 可取 $Q_2 \in \operatorname{Spec} \tilde{B}$, 使得 $\tilde{Q}_2 \cap B = Q_2$. 如果对 $A \subset \tilde{B}$ 的情形已经证明了下降定理, 则可取 $\tilde{Q}_1 \in \operatorname{Spec} \tilde{B}$, 使得 $\tilde{Q}_1 \subset \tilde{Q}_2$, 且 $\tilde{Q}_1 \cap A = P_1$. 这样取 $Q_1 = \tilde{Q}_1 \cap B$ 即可.

(4) 可设 $L/K$ 为可分扩张. 理由为: 设 $\operatorname{char} K = p > 0$, 由于已经假设 $L/K$ 为有限正规扩张, 域扩张 $L/K$ 可以分解为 $K \subset L_{insep} \subset L$, 使得 $L_{insep}/K$ 为有限纯不可分扩张, $L/L_{insep}$ 为有限可分扩张 ([14, Lemma 030M]). 设 $A_{insep}$ 为 $A$ 在 $L_{insep}$ 中的整闭包. 对 $j = 1, 2$, 令 $\tilde{P}_j \coloneqq \{x \in A_{insep} \mid \exists\, n \ge 1, x^{p^n} \in P_j\}$. 则 $\tilde{P}_j$ 为 $A_{insep}$ 中唯一的满足 $\tilde{P}_j \cap A = P_j$ 的素理想, 并且 $\tilde{P}_1 \subset \tilde{P}_2$, $Q_2 \cap A_{insep} = \tilde{P}_2$. 如果对可分扩张 $L/L_{insep}$ 已经证明了下降定理, 则可找到 $Q_1 \in \operatorname{Spec} B$, 使得



B Q1 Q2, Q2'

A P1 P2



$Q_1 \cap A_{insep} = \tilde{P}_1$, $Q_1 \subset Q_2$. 这样 $Q_1$ 也满足 $Q_1 \cap A = P_1$ 的要求.

(5) 由前面几步的约化, 我们可以假设 $L/K$ 为有限 Galois 扩张, $B$ 为 $A$ 在 $L$ 中的整闭包. 令 $G = \operatorname{Gal}(L/K)$ 为 Galois 群. 由上升定理, 可找到 $Q_1' \in \operatorname{Spec} B$ 使得 $Q_1' \cap A = P_1$, 又可找到 $Q_2' \in \operatorname{Spec} B$, 使得 $Q_2' \cap A = P_2$, 且 $Q_1' \subset Q_2'$. 我们断言存在 $g \in G$, 使得 $gQ_2' = Q_2$. 这是因为, 假设 $\forall\, g \in G$, $Q_2 \not\subseteq gQ_2'$, 则由素避引理 3.2.1, 可找到 $x \in Q_2$, 使得 $x \notin gQ_2'$, $\forall\, g \in G$. 这样得到 $y \coloneqq \prod_{g \in G} gx \notin Q_2'$. 由于 $y \in L^G = K$, $y$ 在 $A$ 上整且 $A$ 为整闭整环, 知 $y \in A$. 从而 $y \in Q_2 \cap A = P_2 \subset Q_2'$. 这与 $y \notin Q_2'$ 矛盾. 故存在 $g \in G$, 使得 $Q_2 \subseteq gQ_2'$. 由于 $gQ_2' \cap A = P_2 = Q_2 \cap A$, 以及整扩张时每个纤维中的素理想没有包含关系, 我们得到 $Q_2 = gQ_2'$. 这样取 $Q_1 = gQ_1'$ 即满足条件. □

$(A, m)$ Noetherian 局部环

$\delta(A) =$ A的极小参数子元素个数.

目标: $\delta(A) = \dim A$

Step 1: $\dim A \geq \delta(A)$.

对 $\delta(A)$ 归纳.

$\delta(A) = 0 \Rightarrow m^k = 0$.

$\Rightarrow \dim A = 0$

$\delta(A) \geq 1$, 取 $x_1 \in m$, $x_1$不在极小素理想中.

提升

$$\delta(A/(x_1)) \geq \delta(A) - 1$$

$$\dim(A/(x_1)) \leq \dim A - 1.$$

$$\Rightarrow \quad \dim A - 1 \geq \delta(A) - 1$$

$\square$

Krull 主理想定理 ⇒ Krull 高度定理.

$d(A)$

$i \to l(A/m^i)$

Jordan-Holder thm.

模的长度：合成列.

$$M = M_0 \supseteq \cdots \supseteq M_n = 0$$

$$\ell(M) = n.$$

$$\ell(A/m^i) = \sum_{j=0}^{\infty} \dim_k \frac{m^j(A/m^i)}{m^{j+1}(A/m^i)}$$

$$\boxed{\begin{array}{l} 0 \to M_1 \to M_2 \to M_3 \to 0 \\ \ell(M_2) = \ell(M_1) + \ell(M_3) \end{array}}$$

$$A/m^{i+1} \twoheadrightarrow A/m^i$$

$$\Rightarrow \ell(A/m^{i+1}) \geq \ell(A/m^i)$$

Lemma. $(A,m)$ Noetherian local

ring $\Rightarrow \exists N_0 \geq 1$, s.t. $\exists F(x) \in \mathbb{Q}[X]$, s.t.

$$\forall i \geq N_0 \quad \ell(A/m^i) = F(i)$$

$$\| $$

$$\sum_{j=0}^{i-1} \ell(m^j/m^{j+1}).$$

$$\bigoplus_{j=0}^{+\infty} \frac{m^j}{m^{j+1}}$$ 分次环.

$d(A) := \deg F.$

例.

$k[x,y] \quad (x,y)=m$

$A = k[x,y]_m$

$$A/m^i = \left(k[x,y]/m^i\right)_m$$

$$= k[x,y]/m^i$$

$$l\left(k[x,y]/m^i\right) = \dim_k\left(k[x,y]/m^i\right)$$

$$= \dim_k\left(\{f \in k[x,y] \mid \deg f \leq i-1\}\right)$$

$$= \frac{i(i-1)}{2}$$

附注: $d(A) = \delta(A) = \dim A$.

$\dim A \le d(A) \le \delta(A) \quad (\le \dim A)$.

Step 2. $x \in m \quad \bar{A} = A/(x)$

$d(A),\ d(\bar{A})$ ?

$$\bar{A}/\bar{m}^i = \frac{A/(x)}{(m^i+(x))/(x)} = \frac{A}{m^i+(x)}$$

$$= \frac{A/m^i}{(m^i+(x))/m^i}$$

$$0 \longrightarrow \frac{m^i+(x)}{m^i} \longrightarrow A/m^i \longrightarrow \bar{A}/\bar{m}^i$$

$$\frac{m^i+(x)}{m^i} = \frac{(x)}{m^i \cap (x)} \twoheadrightarrow \frac{Ax}{m^{i-i_0}x}$$

$$(x) \cap m^i = m^{i-i_0}(m^{i_0} \cap (x))$$

$$\subseteq m^{i-i_0}x$$

(Artin - Rees).

Lemma.

$$A \longrightarrow A \qquad \text{为单同态} (x \in m) \text{时}$$

$$y \longrightarrow xy$$

$$\boxed{d(\bar{A}) \leq d(A)-1}$$

$$\ell(\bar{A}/\bar{m}^i) = \ell(A/m^i) - \ell\left(\frac{m^i + (x)}{m^i}\right)$$

$$\leq \ell(A/m^i) - \ell\left(\frac{A}{m^{i-i_0}}\right)$$

$$= F(i) - F(i-i_0)$$

$$P_0 \subsetneq P_1 \subsetneq \cdots$$

$$\begin{array}{c} \Psi \\ (x) \end{array}$$

$$\Rightarrow \dim \bar{A} \geq \dim A - 1$$

$$\Rightarrow \dim A \leq d(A)$$

(3) $d(A) \le \delta(A)$

$x_1 \sim x_s$ 极小参数子.

$l(A/m^i)$

$$A/(x_1,\cdots,x_s)^i \twoheadrightarrow A/m^i$$

$$l(A/m^i) \le l(A/m^i)$$

$$= \sum_{j=0}^{i-1} l\left(\frac{(x_1,\cdots,x_s)^j}{(x_1,\cdots,x_s)^{j+1}}\right)$$

$$\le \left(l\left(\frac{A}{x_1,\cdots,x_s}\right)\right) \sum_{j=0}^{i-1} N_j$$

$+\alpha^{n}$.

$$A/m^n \twoheadrightarrow A/(x_1,\cdots,x_s)$$

$$n_j = \#\{x_1^{a_1}\cdots x_s^{a_s} \mid \sum_{j=1}^{s} a_j \le i-1\}$$

---

Tools from homological algebra

$\mathrm{Tor}_i^A(M,N)$ $\mathrm{Ext}_A^i(M,N)$

A-module complex

$$\to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2}$$

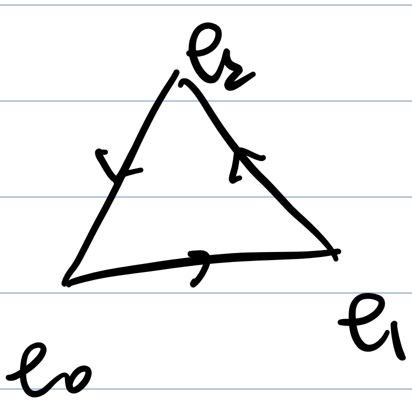
$M_i$ $A$-module $\quad d_i d_{i+1} = 0$

$(M_\bullet, d_\bullet)$ or $(M_\bullet)$

Exact at $M_i$ :

$\operatorname{Im} d_{i+1} = \ker d_i$

$M_\bullet$ Exact if it is exact at each $M_i$




$\partial \langle e_0 e_1 e_2 \rangle = \langle e_0 e_1 \rangle + \langle e_1 e_2 \rangle + \langle e_2 e_0 \rangle$

$= \langle e_1 e_2 \rangle - \langle e_0 e_2 \rangle + \langle e_0 e_1 \rangle$

Čech complex.

---

$$C_n(\Delta_I) \xrightarrow{\partial_n} C_{n-1}(\Delta_I) \xrightarrow{\cdots} \cdots$$

$$C_n(\Delta_I) = \left\{ \langle i_0, \cdots, i_n \rangle \,\middle|\, i_0, \cdots, i_n \in I \right\}$$

$$\langle i_0, \cdots, i_n \rangle \xrightarrow{\partial_n} \sum_{j=0}^{n} (-1)^j \langle i_0 \cdots \hat{i_j} \cdots i_n \rangle$$

$$(C_\cdot(\Delta_I), \partial_\cdot)$$

$$C_n'(\Delta_I) = C_n(\Delta_I) / F_n$$

$F_n$ 由 $\{ \langle i_0 \cdots i_n \rangle - (-1)^{\mathrm{sgn}(\sigma)} \langle i_{\sigma(0)} \cdots i_{\sigma(n)} \rangle \}$

生成子模.

$\langle C'_\cdot(\Delta_I), \partial_\cdot \rangle$

$C''_n(\Delta_I) := \{ \langle i_0 \cdots i_n \rangle \mid i_0 < i_1 < \cdots < i_n \}$

生成的自由 Z-模

$\langle C''_\cdot(\Delta_I), \partial_\cdot \rangle$

$C''_\cdot(\Delta_I)$

$\downarrow$

$$\begin{tikzcd}
C_\cdot(\Delta_I) \arrow[d] \arrow[d, bend right=60] \\
C'_\cdot(\Delta_I)
\end{tikzcd}$$

**Theorem.** $\forall n \in \mathbb{Z}$

$$H_n(C_\cdot(\Delta_I)) = \frac{\ker \partial_m}{\operatorname{Im} \partial_{m-1}} = 0$$

$$H_n(C'_\cdot(\Delta_I)) = 0$$

$$\begin{tikzcd}
C_\cdot \arrow[d, "f"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C_{n-2} \arrow[r] & \cdots \\
D_\cdot & D_n \arrow[r] & D_{n-1} & \cdots\cdots &
\end{tikzcd}$$

f 诱导

$f_*: H_n(C.) \to H_n(D.)$

$[\alpha] \to [f_n(\alpha)]$

f 称为拟同构 (quasi-isomorphism)

若 $f_*: H_n(C.) \to H_n(D.)$

两个复形映射的同伦.

$f: C. \to D.$

$g: C. \to D.$

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

$g$ $f$ $h_n$ $g_n$ $f_n$ $h_{n-1}$ $g_{n-1}$ $f_{n-1}$

$g_{n+1}\ \ f_{n+1} \qquad\qquad\qquad\qquad f_{n-1}$

$$D_{n+1} \xrightarrow{d'_{n+1}} D_n \xrightarrow{d'_n} D_{n-1}$$

若 $\forall n$, 给

$h_n: C_n \to D_{n+1}$, s.t. $\forall n$

$$f_n - g_n = h_{n-1} \circ d_n + d'_{n+1} \circ h_n$$

称 $f$ 与 $g$ 同伦, $f \sim g$

$(f - g = h \circ d + d \circ h)$

命题. $f \sim g$

$\Rightarrow \ \forall n.\ f_*, g_*: H_n(C_\cdot) \to H_n(D_\cdot)$

$f_* = g_*$

Pf: $\Leftrightarrow (f-g)_* = 0$

✓.

Corollary.

若 $Id \sim 0$, $H_n(C_*) = 0$

定义. $f: C. \to D.$

$g: D. \to C.$

若 $f \cdot g \sim Id$

$g \cdot f \sim \mathrm{Id}$

对称 $C_\cdot, D_\cdot$ 同伦等价

Cor. $C_\cdot \sim D_\cdot$

$\Rightarrow H_n(C_\cdot) \xrightarrow{\sim} H_n(D_\cdot)$

命题.

$$H_n(C_\cdot(\Delta_I)) = \frac{\ker \partial_n}{\mathrm{Im}\, \partial_{n-1}} = 0$$

$$H_n(C'_\cdot(\Delta_I)) = \frac{\ker \partial_n}{\mathrm{Im}\, \partial_{n-1}} = 0$$

$C_\cdot(\Delta_I) \sim C'_\cdot(\Delta_I)$

$$\begin{tikzcd}
C_{n+1}(\Delta_I) \arrow[r] & C_n(\Delta_I) \arrow[r] \arrow[dl, "h_n"'] \arrow[d, "Id"] & C_{n-1}(\Delta_I) \arrow[dl, "h_{n-1}"'] \\
C_{n+1}(\Delta_I) \arrow[r, "\partial"'] & C_n(\Delta_I) \arrow[r, "\partial"'] & C_{n-1}(\Delta_I)
\end{tikzcd}$$

找 $h$ s.t. $\partial h + h\partial = Id$

固定 $i \in I$

$$h_n(\langle i_0 \cdots i_n \rangle) = \langle i\, i_0 \cdots i_n \rangle$$

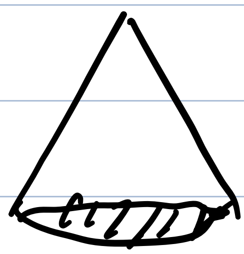
$$h_{-1}(1) = \langle i$$

$$\partial(\langle i\, i_0 \cdots i_n \rangle) + h \circ \partial(\langle i_0 \cdots i_n \rangle)$$

$$= \langle i_0 \cdots i_n \rangle - \sum_{j=0}^{n} (-1)^j \langle i\, i_0 \cdots \hat{i_j} \cdots i_n \rangle$$

$$+\sum_{j=0}^{n}(-1)^j\langle i\ i_1\cdots\hat{i_j}\cdots i_n\rangle$$

$$=\langle i_0\cdots i_n\rangle$$



$h$: 加顶点.

类似 $H_n(C_\cdot(\Delta_I))=0$.

$$C_\cdot \rightsquigarrow C_\cdot\otimes_{\mathbb{Z}} M$$

$M$ 为 $\mathbb{Z}$ 模

$$C_n\otimes_{\mathbb{Z}} M\xrightarrow{d_n\otimes 1}C_{n-1}\otimes_{\mathbb{Z}} M$$

$$C_\cdot \rightsquigarrow \mathrm{Hom}_{\mathbb{Z}}(C_\cdot, M)$$

$$\mathrm{Hom}(C_n, M) \longleftarrow \mathrm{Hom}(C_{n-1}, M)$$

性质. $C_\cdot$ 上若 $\mathrm{Id} \sim 0$

对 $C_\cdot \otimes M$, $\mathrm{Hom}(C_\cdot, M)$ 上均有

$\mathrm{Id} \sim 0$

---

$\mathcal{F}$ 为拓扑空间上预层. 是指

$U \to \mathcal{F}(U)$ ……

Čech 复形.

设 $\{U_i \mid i\in I\}$ 为 $X$ 开覆盖

$\;\;\| $

$\;\;\mathcal{U}$

informal

$$C^{\cdot}(\mathcal{U}, I) := \mathrm{Hom}(\underline{C_{\cdot}(\Delta_I)}, \mathcal{F})$$

$$\|$$

$\mathrm{Hom}(\underset{\sim}{C_n(\Delta_I)}, \mathcal{F}(\Delta_I))$

$$\prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathcal{F}(U_{i_0\dots i_n})$$

$$C^n(\mathcal{U}, \mathcal{F}) = \prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathcal{F}(U_{i_0\dots i_n})$$

$$U_{i_0\dots i_n} = U_{i_0}\cap\dots\cap U_{i_n}$$

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{i_0 \in I} \mathcal{F}(U_{i_0})$$

$$C^{-1}(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$$

$$\begin{array}{ccc} C^n(\mathcal{U}, \mathcal{F}) & \xrightarrow{\delta_n} & C^{n+1}(\mathcal{U}, \mathcal{F}) \\ \| & & \| \\ \prod_{(i_0 \cdots i_n)} \mathcal{F}(U_{i_0 \cdots i_n}) & & \prod_{(i_0 \cdots i_{n+1})} \mathcal{F}(U_{i_0 \cdots i_{n+1}}) \\ \psi & & \end{array}$$

$$s = (s_{i_0 \cdots i_n})_{(i_0, \cdots, i_n) \in I^{n+1}} \xrightarrow{\delta_n} \delta_n(s)$$

$\delta_n(s)$ 的 $(i_0 \cdots i_{n+1})$ 分量为

$$\sum_{j=1}^{n+1} (-1)^j S_{(i_0 \cdots \hat{i_j} \cdots i_{n+1})} \Big|_{U_{i_0} \cdots U_{i_{n+1}}}$$

$$0 \to \mathcal{F}(X) \to C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F}) \to \cdots$$

$$C^1(\mathcal{U}, \mathcal{F}) = \prod_i \mathcal{F}(U_i) \to \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j) = C^2(\mathcal{U}, \mathcal{F})$$

$$\begin{array}{ccc} C^{n+1}(\mathcal{U}, \mathcal{F}) & \xrightarrow{?} & C^n(\mathcal{U}, \mathcal{F}) \\ \| & & \| \\ \prod \mathcal{F}(U_{i_0 \cdots i_{n+1}}) & & \prod \mathcal{F}(U_{i_0 \cdots i_n}) \end{array}$$

性质. 设 $\exists i \in I$, s.t. $U_i = X \in \mathcal{U}$

则 $\check{H}^n(\mathcal{U}, \mathcal{F}) = 0$, $\forall n$

**证明** 设 $U_i = X$. 对 $n \geq 0$, 我们定义如下同态:

$$h^{n+1} : C^{n+1}(\mathcal{U}, \mathcal{F}) \longrightarrow C^n(\mathcal{U}, \mathcal{F}) \tag{6.1–5}$$

$$s = (s_{i_0 \dots i_{n+1}}) \longmapsto h^{n+1}(s) \tag{6.1–6}$$

其中 $h^{n+1}(s)$ 的 $(i_0 \dots i_n)$ 分量为 $h^{n+1}(s)_{i_0 \dots i_n} := s_{ii_0 \dots i_{n+1}} \in \mathcal{F}(U_{i_0 \dots i_n})$. 注意 $s_{ii_0 \dots i_{n+1}}$ 虽然是 $\mathcal{F}(U_{ii_0 \dots i_n})$ 中的元素, 但是因为 $U_{ii_0 \dots i_n} = U_i \cap U_{i_0} \cap \cdots \cap U_{i_n} = X \cap U_{i_0} \cap \cdots \cap U_{i_n} = U_{i_0 \dots i_n}$, 我们将 $s_{ii_0 \dots i_n}$ 自然看作 $\mathcal{F}(U_{i_0 \dots i_n})$ 中的元素.

再定义同态 $h^0 : C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^{-1}(\mathcal{U}, \mathcal{F})$, $s = (s_{i_0}) \longmapsto s_i$. 通过直接验证可以看到 $h^{\cdot}$ 给出了 Čech 复形 $C^{\cdot}(\mathcal{U}, \mathcal{F})$ 上恒等同态与零同态的一个同伦. 从而得到 Čech 复形的零调性. □

$A$ 为环, $M$ 为 $A$ 模

$\mathcal{F} = \widetilde{M}$

$$D(f) \longmapsto M_f$$

$$D(g) \longmapsto M_g$$

命题: $U_i = D(f_i)$ $i=1,\cdots,n$ 为

Spec A 的有限开覆盖, 记为 $\mathcal{U}$

则 $H^n(\mathcal{U}, \tilde{M})=0$

证明: $H^n(\mathcal{U}, \tilde{M})=0$

$\Leftrightarrow \forall i,\ H^n(\mathcal{U}|_{D(f_i)}, \tilde{M}_{f_i})=0$

由上述性质即知.

注: 有限性用在直积与局部化交换.

此为 $\check{C}ech$ 复形的相似性,

即限制的 Čech 复形

推论. $\tilde{M}$ 为层.